The Ross Recovery Theorem and The Term Structure of Interest Rates

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ABSTRACT

This paper presents the importance of a risk-free rate condition in Ross recovery estimation (Ross (2015)). The term structure of risk-free rates explains the differences between approaches of Ross recovery estimations, such as Ross Stable (Jackwerth & Menner (2020)) and the original Ross recovery. A flat term structure of risk-free rates results in a Ross recovered probability distribution identical to the risk-neutral probability distribution in Ross recovery. After considering risk-free rates with a market example, empirical evidence still shows Ross recovered probability distribution is close to the risk-neutral probability distribution. Besides, this paper presents some challenges with Ross recovery empirical applications. Ross recovery with a short transition time implies a nonnegative matrix root for the transition matrix with a long transition time. Different least squares estimations are not equivalent when there is no unique and accurate fitting with the market state prices. A sparse spot state price surface probably results in a relatively stable pricing kernel in Ross recovery.

Keywords: Ross recovery; Term structure of interest rates; Stochastic discount factor; Risk-neutral probability.

JEL classification: G13

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1 Introduction

Understanding the expected physical probability distribution of future asset prices is important for all aspects of finance. Estimation methods of risk-neutral probability distribution have already been well developed, such as Jackwerth (2004) and Figlewski (2010). Empirical analysis shows risk-neutral probability distribution contains important market information. However, risk-neutral probabilities are adjusted by market investors' risk preferences. It is still difficult to recover the physical probabilities from the risk-neutral probabilities without specifying a representative agent's utility function. Ross (2015) tries to solve the problem and proposes a theorem to recover the physical probability distribution with just a snapshot of option prices. However, Borovička et al. (2016) argue that instead of recovering physical probabilities, Ross recovery theorem implicitly assumes the martingale component of stochastic discount factors as identical to unity and recovers long-term risk-neutral probabilities. Empirical evidence also shows that the implicit assumption of the martingale component in Ross recovery is not correct for recovering physical probabilities (Bakshi et al. (2018)). Although Ross recovered probabilities are not physical probabilities, it is still unclear whether Ross recovered probabilities provide additional information in comparison to risk-neutral probabilities. Audrino et al. (2019) shows Ross recovered moments yield additional predictive information to risk-neutral moments while Jackwerth & Menner (2020) find Ross recovered probability distributions fail to predict future returns or realized volatility. Besides, based on different estimation approaches to the Ross recovery theorem, Jackwerth & Menner (2020) present completely different recovered probabilities from the same spot state prices. A reliable and consistent Ross recovery estimation process is needed in empirical studies. Despite the debate on the empirical application for the Ross recovery theorem, Martin & Ross (2019) theoretically present an important relationship between Ross recovery and the term structure of risk-free rate. This paper contributes to the discussion of Ross recovery empirical applications by incorporating the market interest rates in Ross recovery estimation.

This paper shows the importance of risk-free rates for Ross recovery. Ross recovered probabilities are sensitive to the term structure of risk-free rates. Existing Ross recovery estimations omit risk-free rates and are thus ill-conditioned. The original Ross recovery and extended versions of Ross recovery estimations provide significantly different recovered probabilities from risk-neutral probabilities. However, Generalized recovery (Jensen et al. (2019)) and Ross Stable (Jackwerth & Menner (2020))[1] yield a relatively stable pricing kernel, indicating only a small difference between the risk-neutral measure and Ross recovery measure. This paper links and explains previous empirical differences between the original Ross recovery estimation and Generalized recovery by the term structure of risk-free rates.

[1] This paper uses Generalized recovery and Ross Stable interchangeably as they independently propose this recovery estimation.

Incorporating the risk-free rate condition provides a reliable and consistent estimation process for the Ross recovery theorem. Ross recovered probabilities are the same as the risk-neutral probabilities when the term structure of risk-free rates is flat. In real-world practice, when the risk-free rate term structure is not flat, Ross recovery estimation with a risk-free rate condition yields recovered probabilities close to the risk-neutral probabilities like Generalized recovery estimation.

Besides, this paper presents three other challenges in Ross recovery estimations. First, Ross recovery estimations with different lengths of one-period transition time are not equivalent. Ross recovery with a short period transition implies a condition of the existence of nonnegative n^{th} root of Ross recovery with a long period transition. Empirical applications with a long-term transition time are more flexible with the transition matrix during the estimation process. Second, different least squares representations are not equivalent when there is no unique and accurate transition matrix for the market state prices. Linear and nonlinear representations of Ross recovery imply significantly different term structures of risk-free rates. Based on this idea, this paper further proposes a multi-period Ross recovery estimation so that the fitted state price surface in Ross recovery is correctly reflected. Third, the number of states in the transition matrix also plays an important role in Ross recovery estimation. Ross recovery results in more stable and well-behaved recovered probabilities with sparse states in comparison to dense states for the same number of expiries.

This paper also connects to the non-parametric Risk-Neutral Distribution (RND) estimation developed by Figlewski (2010). The non-parametric method provides an unbiased spot state price surface in comparison to other parametric risk-neutral distribution methods. This paper extends Figlewski's non-parametric estimation into different horizons and makes the term structure of risk-free rates correctly reflected in the risk-neutral probability surface. A smooth and accurate risk-neutral probability surface is the prerequisite for Ross recovery estimation.

The rest of this paper is organized as five parts. Section (2) shows the spot state price surface estimation which is the prerequisite for the Ross recovery theorem. Section (3) reviews the Ross recovery theorem and presents the empirical estimation for the Ross recovery theorem. It brings two challenges Ross recovery, the term structure of interest rates and the length of one-period transition time. Section (4) is the main discussion of this paper that by incorporating the interest rate curve into Ross recovery, recovered probabilities are stable and close to risk-neutral probabilities. Section (5) is the nonlinear multi-period Ross recovery estimation. Section (6) compares the Ross recovery implied information from different approaches.

2 Spot State Price Surface

A state price $\pi(t_0, t_0+t, i, j)$, or Arrow-Debreu price[2], is the expected price at time t_0 for a unit payment at time $t_0 + t$ when the economy transforms from one state i to another state j accordingly. Using spot market asset prices, especially a cross of option prices with a series of strikes, the spot state price $\pi_0(t, j)$ is determined by the state j in horizon t given a spot state $i = i_0$ at spot time $t_0 = 0$. The spot state price estimation is independent of the Ross recovery theorem. But a reliable and well behaved spot state price surface is the prerequisite for Ross recovery estimation as the transition matrix in Ross recovery theorem is directly estimated from the spot state price surface.

This paper uses Figlewski (2010) non-parametric risk-neutral distribution to estimate spot state prices at different expiries. It also makes some adjustments for Figlewski's method to ensure that the risk-neutral distribution accurately reflects the term structure of interest rates and the state prices are fully extracted in both moneyness and time-to-maturity dimensions.

2.1 Data

This paper uses the S&P 500 weekly options (SPXW) on January 4, 2019, as an example for empirical analysis. But the estimation process can be easily extended to other assets or dates. S&P 500 weekly European put and call option quotes are extracted from Thomson Reuters DataScope. The last midquote on each date is used as the option close price. S&P 500 weekly option has more expiry dates[3] than the standard S&P 500 options so that a more detailed S&P 500 state price surface is possible. The moneyness of each S&P 500 option contract is the strike price divided by the corresponding S&P 500 index level.

U.S. risk-free rate is the zero-coupon yield from OptionMetrics. A cubic spline interpolation[4] is applied to the zero-coupon bond price which is based on the zero-coupon yield. The risk-free rate with the same time to maturity as the option contract is estimated from the interpolated zero-coupon bond price curve. Daily S&P 500 index dividend yields are also from OptionMetrics. The dividend ratio is the average realized dividend yields from the trading date to the expire date of each option contract.

2.2 Risk-Neutral Distribution Surface

This paper excludes in-the-money option contracts that are call options with moneyness lower than 99.5% and put options with moneyness higher than 100.5%. Based on the standard option pricing model of Black & Scholes (1973), at-the-money and out-of-the-money option prices are transformed into Implied Volatilities

[2] This paper uses state price and Arrow-Debreu price interchangeably
[3] SPXW options provide the weekly Monday, Wednesday, Friday expiry dates
[4] The cubic spline interpolation is conducted by MATLAB function *spaps*.

(IVs). A discrete IV surface consists of option IVs with their moneyness and time to maturity. A fourth degree spline interpolation on both moneyness and time to maturity[5] is conducted on the discrete IV surface to get a smooth fitted IV surface $\mathbf{IV} = [IV(X_j, T)]$. The interpolation is from 1 day to 252 days with a 1-day interval and from moneyness 0 to moneyness 2 with an interval of 0.0001. The extrapolated IV beyond the minimum or maximum market moneyness is excluded in the next step to avoid any unreliable shape in the deep tails. The fitted IV surface is transformed back to the call option price surface $\mathbf{C} = [C(X_j, T)]$ by the Black-Scholes model again. The risk-neutral probability density $f(X, T)$ and cumulative risk-neutral probability $F(X, T)$ at strike price X_j and time to maturity T is estimated by the approximation formula of Breeden & Litzenberger (1978)

$$f(X_i, T) \approx e^{rT} \frac{C_{i+1,T} - 2C_{i,T} + C_{i-1,T}}{(\Delta X)^2} \tag{1}$$

and the cumulative probability function is

$$F(X_i, T) \approx e^{rT} [\frac{C_{i+1,T} - C_{i-1,T}}{X_{i+1,T} - X_{i-1,T}}] + 1 \tag{2}$$

The above procedures provide a central risk-neutral probability distribution from the minimum market strike price to the highest market strike price. The tails of the risk-neutral probability distribution are estimated with Generalized Extreme Value (GEV) distribution in the form of

$$G_(x) = exp[-(1 + \xi \frac{S_T - \mu}{\sigma})^{\frac{1}{\xi}}] \tag{3}$$

where ξ determines the shape, μ is the mean, σ is the variance of GEV model.

The six parameters of the left and right tails are estimated together by satisfying the following constraints: i) the density of the GEV tail equals the density of the central RND at the first and the second connection points in both left and right tails. The first connection point of the right tail is the strike at the 97% cumulative risk-neutral probability or the maximum market strike whichever is smaller. The percentile of the first right connection point is α_R The second connection point is the strike that is closest to the percentile $\alpha_R - 3\%$. The first connection point of the left tail is the strike at the 3% cumulative risk-neutral probability or the minimum market strike whichever is greater. The percentile of the first left connection point is α_L. The second left connection point is the strike that is closest to the percentile α_L+3%. ii) The sum of central RND and tail probability equals 100%. iii) The expectation of moneyness based on central density and tail distribution equals corresponding risk-neutral measure, the risk-free rate return minus the dividend ratio as

[5]The spline interpolation uses *scipy.interpolate.bisplrep* function in Python package *scipy* version 1.4.1

$exp((r-q)T)$.

The procedures above provide a smooth and well-behaved RND surface. Figure (1) panel A shows that the RND surface estimated on January 4, 2019 is smooth and reliable. Figure (1) panel B shows the term structure of RND from short to long horizons. Short-term RND is more concentrated around at-the-money and relatively symmetric. Long-term RND is more disperse and negatively skewed.

Insert Figure (1)

Figure (2) compares the implied risk-free rates from the RND surface and market risk-free rates. The market risk-free rate is from 2.4% to 2.8% for horizons within one year. It increases sharply in the short term and decreases slowly in the medium and long terms as the horizon increases. The RND implied risk-free rate is consistent with the market risk-free rate indicating a reliable estimation based on the risk-neutral measure.

Insert Figure (2)

The number of distinct economy states determines the number of strikes or moneyness in RND that further determines the size of unknown parameters in Ross recovery. To have a relatively good balance between the smoothness of state price surface and the number of unknown parameters in the following Ross recovery theorem, this paper applies 100 states from moneyness 0 to moneyness 2 with an interval of 0.02. The horizon is from 2 days to 250 days with an interval of 2 days. The RND surface is reduced into a 125×100 RND matrix $\mathbf{Q} = [q(t,i)]$ where t is the horizon of the RND and i is the moneyness. Each row in RND matrix $\mathbf{Q}$ is an RND with time to maturity as t. Each column in the RND matrix $\mathbf{Q}$ shows the risk-neutral probability at state i. In the later part of this paper, the economy states are further reduced into 20 states with an interval of 0.1 moneyness because of time-consuming nonlinear calculation.

2.3 Risk-Neutral Distribution to State Price

The risk-neutral probability matrix $\mathbf{Q}$ and the spot state price matrix $\mathbf{\Pi} = [\pi_0(t,i)]$ can be transformed to each other with an interest rate curve. Risk neutral probability $q(t,i)$ is transformed to spot state price $\pi_0(t,i)$ directly by corresponding risk-free rate r_f^t. With continuous compounding,

$$\pi_0(t,i) = q(t,i)exp(-r_f^t \cdot t) \tag{4}$$

Besides, the spot state price with horizon $t=0$ equals one for the current state and zero for all other states, i.e. a state price vector Π_0 with only one non-zero entry. The $t=0$ state price vector can be added to the spot state price matrix as the first row to have a complete spot state price matrix $\mathbf{\Pi}$ from time 0 to time T.

3 The Ross Recovery Theorem

This section reviews the Ross recovery theorem and its empirical applications. Based on the spot state price matrix, Ross claims a recovery method to get physical probability distributions and the pricing kernel at the same time. However, Borovička et al. (2016) prove Ross recovered probabilities are not physical probabilities. The long-term risk is still incorporated inRoss recovered probabilities. This paper follows the assumptions in Ross recovery and aims to discuss the information in Ross recovered probabilities. Previous different empirical applications of Ross recovery generate significantly different results. This section explains the differences by exploring the term structure of risk-free rates and the one-period transaction time.

3.1 Original Ross Recovery Theorem

This paper focuses on Ross recovery with finite discrete states. The Ross (2015) recovery theorem explicitly makes three assumptions. First, it assumes a time-homogeneous one-period transition matrix that links the state prices at time t and state prices at time $t+h$ as a Markov chain process. A state price $\pi(t, t+h, i, j)$ with $t > 0$ is the transition price from a future time t to a further time $t+h$. Transition state price is not directly observable in the market. Because Ross recovery assumes the one-period transition matrix is time-homogeneous, independent from time t. The one-period transition state price $\pi(t, t+h, i, j) = a(i, j), \forall t$ is only determined by the start state i and end state j given a transition period h. Based on state prices at time t and the transition matrix, each state j at time $t+h$ is calculated as:

$$\pi_0(t+h, j) = \sum_{i \in N} \pi_0(t, i) a(i, j), \forall j \in N \tag{5}$$

where N is the set of all states. $a(i, j)$ is the transition state price from state i to state j during any one-period time h and is the entry in the transition matrix $\mathbf{A} = [a(i, j), i, j \in N]$. Because the current spot state prices with horizon $t = 0$ is determined by the current state as Π_0. The one-period state price transition matrix implies state prices at any time $n \cdot h (n \in \mathbb{N})$ as

$$\Pi_n = \Pi_0 \times \mathbf{A}^n \tag{6}$$

where $\Pi_n = [\pi_0(n \cdot h, j)]$ is a row vector indicating the spot state prices from current state to state j at time $n \cdot h$, Π_0 and Π_n are the 0^{th} and n^{th} row in the spot state price surface $\mathbf{\Pi}$.

Second, Ross recovery theorem assumes all transition state prices in $\mathbf{A}$ are positive.

Third, as the stochastic discount factor (m) links the physical probability (p) and state price (π) in the form of $p = \pi/m$, Ross further assumes the stochastic discount factor is transition independent. Then the one-period stochastic discount factor from state i to state j is

$$m_{i,j} = \delta \frac{u_j^{'}}{u_i^{'}} \tag{7}$$

where $u^{'}$ and δ can be interpreted as marginal utility and a utility discount factor respectively. This paper follows the notations of Ross recovery. But it should be aware that instead of recovering risk-neutral probabilities to physical probabilities, the transition-independent stochastic discount factors recover risk-neutral probabilities to long-term risk-neutral probabilities as Borovička et al. (2016) prove.

The one-period state price transition matrix $\mathbf{A}$ corresponds to a one-period Ross recovered probability transition matrix $\mathbf{S}$ based on the above stochastic discount factors. Each row in the one-period Ross recovered probability transition matrix indicates all possible outcomes from a state during one period. Therefore, the sum of each row in the one-period Ross recovered probability transition matrix equals one. Based on this idea and a little representation, the state price transition matrix satisfies the following equation:

$$\mathbf{A} \times Z = \delta Z \tag{8}$$

where Z is a vector of the inverse $u_i^{'}$, i.e. $z_i = 1/u_i^{'}$.

Equation (8) is an eigenvalue problem. By using the Perron-Frobenius theorem, the largest eigenvalue corresponds to the only eigenvector with strictly positive z_i. Therefore, given the state price transition matrix, based on the solution of equation (8), the stochastic discount factor and Ross recovered distribution are uniquely determined. The spot state price surface $\mathbf{\Pi}$ can be transformed to a Ross recovered probability surface $\mathbf{P}$ with the stochastic discount factors. Ross recovered probability at time $n \cdot h$ in the Ross recovered probability surface $\mathbf{P}$ is calculated in a similar way as equation (6)

$$P_n = P_0 \times \mathbf{S}^n \tag{9}$$

where P_0, P_n are the 0^{th} and n^{th} row in the Ross recovered probability surface $\mathbf{P}$. As the state price transition matrix directly implies the recovered probability transition matrix, Ross recovery estimation only needs to extract the state price transition matrix from the spot state price matrix.

3.1.1 Ross Recovery Implied Interest Rate

In an economy that follows the Ross recovery theorem, the transition matrix fully determines risk-free rates, including the state risk-free rate, the long rate curve, and the short rate curve.

The sum of each row in the state price transition matrix is the one-period risk-free discount factor in the corresponding state. The one-period state risk-free rate vector of all states is

$$R = [r_i], \text{and}\, r_i = -log(\sum_j a(i,j)) \tag{10}$$

where r_i means the one-period risk-free rate if the economy is at state i, $a(i,j)$ is the one-period state price transition matrix from Ross recovery theorem. Because the transition matrix is time-homogeneous, state risk-free rates R are also independent of time.

Martin & Ross (2019) and Ross (2015) are especially interested in the long end risk-free rate because of the convergence of the transition matrix in infinite horizon. But the long end risk-free rate is not directly observable in the market. This paper uses the current state and the transition matrix to calculate the whole risk-free rate curve. Based on the current spot state price vector Π_0 and the transition matrix $\mathbf{A}$, the N period state prices is calculated as $\Pi_0 \times \mathbf{A}^n$. Then the N period long rate is calculated as

$$y_n = -log(\sum_j \pi_0(n \cdot h, j)), \text{and} \quad \Pi_n = [\pi_0(n \cdot h, j)] = \Pi_0 \times \mathbf{A}^n \tag{11}$$

where y_n is the N period long rate from time 0 to time $n \cdot h$, $\Pi_n = [\pi_0(n \cdot h, j)]$ is the n period Ross recovery implied spot state prices which is calculated by formula (6).

The short rate is the forward interest rate from a future time to a further time. There are two ways to calculate the short rate. First, the short rate can be inferred from the long rate curve in equation (11). Second, the short rate can be also estimated from the state interest rate in equation (10). Besides, the transition state price uniquely determines a transition Ross recovered probability matrix shown as equation (9). The n^{th} one-period short rate from time $n \cdot h$ to time $(n+1) \cdot h$ is calculated as

$$b_n = -log(P_n \cdot R) = -log(\sum(P_0 \times \mathbf{S}^n) \cdot R) \tag{12}$$

where P_n is the n^{th} period Ross recovered probability as formula (9), R is the one-period state risk-free rate as formula (10).

3.2 Ross Basic Estimation

In the original estimation by Ross (2015) and the Ross Basic estimation labeled by Jackwerth & Menner (2020), the state price transition matrix is estimated by minimizing the one-period transition errors. Following the Ross Basic approach of Jackwerth & Menner (2020), this paper uses the overlapping one-period transition data. The spot state price surface $\mathbf{\Pi}$ from day 0 to day 250 with an interval of 2 days as estimated in section (2) is rearranged into two subgroups of spot state price surfaces $\mathbf{\Pi}_a$ and $\mathbf{\Pi}_b$. Spot state price surface $\mathbf{\Pi}_a$ is from day 0 to day 230. Spot state price surface $\mathbf{\Pi}_b$ is from day 20 to day 250. The one-period transition is 20 business days (one month). The 20-day transition state price matrix $\mathbf{A}$ links the state prices at day t with state prices at day t+20 as $\mathbf{\Pi}_b = \mathbf{\Pi}_a \times \mathbf{A}$. This equation is solved by minimizing the following least squares estimation.

$$\min_{a(i,j)} \sum_{j \in N} \sum_{t=0}^{230} (\pi_0(t+20, j) - \sum_{i \in N} \pi_0(t,i) a(i,j))^2 \quad s.t. \quad a(i,j) \geq 0 \tag{13}$$

where $\pi_0(t,i)$ and $\pi_0(t,j)$ are the spot state prices in the spot state price surface $\mathbf{\Pi}$. This least squared problem is estimated by *lsqnonneg* function in MATLAB[6].

Figure (3) shows the state price transition matrix by Ross Basic estimation. In Figure (3) Panel A of the complete transition matrix of Ross Basic, there is an extreme state price with state moneyness close to 0. Because market spot state prices with moneyness below 0.4 or above 1.4 are close to 0. They correspond to the deep tails of risk-neutral probability distributions. There is little information for the estimation of transition matrix with transition start state below moneyness 0.4 or above moneyness 1.4. Therefore, some extreme values in the rows corresponding to the deep tails of the spot state prices have no obvious impact on the estimation process. Figure (3) Panel B excludes the rows in the transition matrix corresponding to the deep tails and starts from moneyness 0.4 to moneyness 1.4. Similar to the results of Jackwerth & Menner (2020), the main diagonal of the transition matrix has relatively higher state prices as a result of higher probabilities of the same state at the end of the transition period as the initial of the transition period. However, it is difficult to understand the state prices away from the main diagonal. Especially some states have high transition prices while the states next to them have close to 0 transition prices.

Insert Figure (3)

Figure (4) compares the fitted risk-neutral probabilities and recovered probabilities with the market risk-neutral probabilities. The fitted one-period risk-neutral probabilities at time t are calculated by the market risk-neutral probabilities at time $t-h$ and the one-period transition matrix. The multi-period fitted risk-neutral probabilities at time t is based on the spot state prices at current day t_0 multiply by $(t/h)^{th}$ (20 days per period, 6 periods for 120 days in this example) power function of the transition price matrix as formula (6). Recovered probabilities are the multi-period risk-neutral probabilities recovered by the transition matrix implied stochastic discount factors. The fitted one-period risk-neutral probabilities are

[6]Least squared problem (13) can be written in a standard $C \cdot x - d$ form as Appendix 1.1. *lsqnonneg* function in MATLAB provides a fast and reliable solution to the standard $C \cdot x - d$ linear least squares problem.

not necessarily the same as the Ross recovery implied multi-period risk-neutral probabilities in Ross recovery because of estimation errors.

This Ross Basic estimation fits well with the market spot state prices. There is no obvious difference between the fitted one-period risk-neutral probabilities and the market risk-neutral probabilities for the 20-day horizon. However, in the long term, there are some minor differences around the mode of the distributions. Long-term fitted risk-neutral probabilities based on a one-period transition present small differences from the market risk-neutral probabilities while fitted risk-neutral probabilities based on multi-period transition show relatively large estimation errors. It raises a challenge that the estimation of minimizing one-period transition errors, as Ross Basic, may not minimize the difference between the Ross recovery multi-period risk-neutral probabilities and the market risk-neutral probabilities. Besides, because the Ross Basic least squared problem (13) is an overspecified linear equation system with a nonnegativity constraint for the transition state price. An exact solution is not possible given the market spot state prices. The least squares estimation provides a satisfying but not the same fitted state price surface as the market state price surface.

Despite only small errors in the Ross Basic estimation, the recovered probability distribution is not smooth or well-behaved. The recovered probability at moneyness 0.94 is negligible while the risk-neutral probability is significant at the same moneyness for the 20-day horizon as shown in Figure (4) Panel A. Long-term 120-day recovered probability distribution has even more kinks. The Ross Basic estimation in (13) only minimizes the one-period transition errors. Long-term recovered probability distribution therefore may have more estimation errors and unreliable shapes.

Insert Figure (4)

Figure (5) shows the Ross transition matrix implied risk-free rate. Panel A shows the state interest rates. Panel B compares the implied with market long rate curves. First, the extreme negative implied state interest rates are unlikely to happen in the market. The Ross Basic estimation results in unreasonable large and negative risk-free rates. Second, the implied long rate curve is far different from the market long rate curve. Ross Basic estimation fails to incorporate the market interest rates that determine both the first moment and risk-free discount factor in spot state price estimation. A spot state price surface provides a solution for least square problem (13). But such a solution is not necessarily consistent with the Ross recovery economy. Therefore, Ross Basic estimation is ill-conditioned as it completely ignores the term structure of interest rates which may be the reason for the dramatic shape of stochastic discount factors.

Insert Figure (5)

Ross (2015) and Jackwerth & Menner (2020) notice the unreasonable state interest rate. They further propose incorporating extra estimation conditions to Ross Basic estimation, such as imposing upper and lower bounds in the transition matrix implied interest rates (Ross Bounded) or assume there is only one mode in each row of the transition matrix (Ross Unimodal). However, those conditions and constraints are not implied in the original Ross Recovery theorem. And different additional constraints result in different transition matrices by the empirical work of Jackwerth & Menner (2020). This paper keeps the original Ross recovery assumptions and argues that the ill-conditioned Ross Basic estimation should incorporate the term structure of interest rates and a multi-period transition process as shown in the later sections.

3.3 Ross Root Estimation

This section brings one challenge in Ross recovery estimation. Different lengths of one-period transition time are likely to result in significantly different recovered transition matrices. Because transition matrix with a short period implicitly has a condition that the power function of the short period transition matrix is the long period transition matrix which further implicitly require such a long term transition matrix has a nonnegative k^{th} root.

3.3.1 The One-period Transition

Based on the assumption in the Ross recovery theorem, there is a time-homogeneous one-period transition matrix. Ross makes no further assumption on the length of one-period transition time. The one-period transition can be one day or one month. However, in the empirical estimations, lengths of one-period transition time imply different conditions.

Like Ross Basic estimation, the spot state price surface $\mathbf{\Pi}$ can be rearranged into two subgroups $\mathbf{\Pi}_a$ and $\mathbf{\Pi}_b$ with a time lag as h. A state price transition matrix $\mathbf{A}_h$ with a transition period h links these two subgroups as $\mathbf{\Pi}_b = \mathbf{\Pi}_a \times \mathbf{A}_h$. Besides, the transition matrix $\mathbf{A}_h$ can be calculated from a state price transition matrix $\mathbf{A}_{h/2}$ with a transition period $h/2$ as $\mathbf{A}_h = \mathbf{A}_{h/2}^2$. Then the two subgroups can be linked by transition matrix $\mathbf{A}_{h/2}$ as $\mathbf{\Pi}_b = \mathbf{\Pi}_a \times \mathbf{A}_{h/2}^2$. Therefore, there are two approaches to estimate the transition matrix given a spot state price surface. The approach of estimating $\mathbf{A}_{h/2}$ is equivalent to estimating $\mathbf{A}_h$ with a condition that $\mathbf{A}_h$ has a nonnegative square toot. According to the uniqueness and existence condition of a nonnegative square root of a nonnegative matrix by (Tam & Huang (2016)), a long period transition matrix $\mathbf{A}_h$ directly estimated from Ross Basic approach may have no nonnegative square root.

The above analysis can be easily extended to k^{th} root of the transition matrix. In Ross Recovery estimations, by choosing a relatively long one-period transition time, the estimation process has more flexibility to fit the market spot state prices. By choosing a relatively short one-period transition time, the estimation process implicitly incorporates the existence of a nonnegative k^{th} root of a long period transition matrix.

3.3.2 Ross Root Estimation

In Ross Basic estimations, the one-period transition time is 20 days. This paper proposes a new estimation method labeled Ross Root with a one-period transition of 2 days. Ross Root estimation with a transition time of 2 days implicitly requires the existence of a nonnegative 10^{th} root for a one-month transition matrix. While the one-month transition matrix in Ross Basic estimation may have no such nonnegative 10^{th} root.

Spot state price surfaces $\mathbf{\Pi}_a$ and $\mathbf{\Pi}_b$ are the same as the Ross Basic estimation. The 2-day transition state price matrix $\mathbf{A}_{2d}$ links the state prices at day t with state prices at day t+20 as $\mathbf{\Pi}_b = \mathbf{\Pi}_a \times \mathbf{A}_{2d}^{10}$. $\mathbf{A}_{2d}^{10}$ has 10^{th} degree of unknown transition state prices which makes it difficult to estimate. Therefore, Ross Root is estimated by the same least squares problem (13) but changing the transition time from 20 days to 2 days.

Figure (6), (7), and (8) are presented in a similar way as Ross Basic. Figure (6) is the state price transition matrix of Ross Root estimation. First, the transition states corresponding to the deep tails of the spot state price surface present extreme values like Ross Basic estimation. Second, the state prices are much higher on the main diagonal of the transition matrix. But the higher state prices on the main diagonal are more significant than the Ross Basic transition matrix. By changing the transition period h, Ross Root provides a significant difference transition matrix to Ross Basic.

Insert Figure (6)

Figure (7) shows recovered distributions from Ross Root estimation. Panel A and B show risk-neutral probabilities and recovered probabilities with 20-day and 120-day horizons. Like Ross Basic estimation, Ross Root estimation fits well with the market spot state prices. In the 20-day horizon, Ross Root fitted risk-neutral probabilities present no obvious difference from the market risk-neutral probabilities. In the 120-day horizon, there are some minor differences between the fitted risk-neutral probabilities and the market risk-neutral probabilities around the mode. Unlike Ross Basic, the fitted risk-neutral probability distributions from one-period and multi-period estimations are not different as the one-month one-period transition matrix is calculated by 10^{th} power function of the 2-day transition matrix. The nonlinearity of long-term Ross fitted risk-neutral probability distribution is reflected in Ross Root estimation.

Although the Ross Root is similar to Ross Basic with risk-neutral probability distributions, the Ross Root recovered probabilities are dramatically different from Ross Basic recovered probabilities. The Ross Root recovered probability distribution is not smooth, indicating stochastic discount factors with dramatic shapes. Because Ross Root implicitly imposing an additional constraint. The one-month transition matrix in Ross Root is less flexible to fit the market state prices. In comparison to Ross Basic, small differences in the fitted risk-neutral probabilities result in significantly different recovered probabilities in Ross Root estimation.

Insert Figure (7)

Figure (8) shows the Ross Root transition matrix implied interest rates. Ross Root implied interest rates have a similar pattern as Ross Basic implied interest rates. First, largely positive and negative interest rates in some states are unlikely to happen in the market, e.g. over 6000% annualized state interest rate. Second, the implied long rates, from -10% to 15%, are far different from the actual market long rates. Ross Root estimation fails to incorporate the market interest rate. It is not a surprise that Ross Root has similar problems as Ross Basic estimation because they are estimated by the same least squares problem. However, it should be noticed that by changing the transition period to a shorter time, the Ross Basic estimation provides significantly different results from the same spot state price surface.

Insert Figure (8)

Existing literature provides no solution to the challenge of lengths of one-period transition time. Following previous literature, this paper focuses on the one-month transition matrix.

3.4 Ross Stable/ Generalized Recovery

In contrast to imposing addition constraints or assumptions to the Ross recovery theorem, Generalized recovery (Jensen et al. (2019)) and Ross Stable (Jackwerth & Menner (2020)) only keep the assumption of transition-independent stochastic discount factors in Ross recovery. The one-period time-homogeneous transition matrix is excluded from Generalized recovery and Ross Stable methods. With the assumption that the stochastic discount factor is transition-independent as the form of formula (7), the spot state price surface can be transformed to a surface of recovered probability. A set of equations without transition matrix is specified as:

$$\begin{bmatrix} \pi_0(1,1) & ... & \pi_0(1,j) & ... & \pi_0(1,N) \\ \pi_0(2,1) & ... & \pi_0(2,j) & ... & \pi_0(2,N) \\ \vdots & \vdots & ... & \vdots & \\ \pi_0(T,1) & ... & \pi_0(T,j) & ... & \pi_0(T,N) \end{bmatrix} \times \begin{bmatrix} z_1/z_1 \\ \vdots \\ z_{k-1}/z_1 \\ z_k/z_1 \\ \vdots \\ z_N/z_1 \end{bmatrix} = \begin{bmatrix} \delta^1 \\ \delta^2 \\ \vdots \\ \delta^{T-1} \\ \delta^T \end{bmatrix} \tag{14}$$

where $\pi_0(t,j)$ is the spot state price in the spot state price surface $\mathbf{\Pi}$, z_k and δ are the inverse of marginal utility and the utility discount factor respectively with the same meaning as the Ross recovery theorem, the estimation in each row of this equation system means the sum of recovered probability equals one. This equation system is estimated by minimizing the squared errors as

$$\min_{z_k,\delta} \sum_{t=1}^{t=250} ((\sum_{k=1}^{N} \pi_0(t,k) z_k/z_1) - \delta^t)^2 \quad s.t. \quad z_k \geq 0, \delta \geq 0 \tag{15}$$

following Jackwerth & Menner (2020), this least squares estimation is labelled as Ross Stable.

Figure (9) shows the recovered probability distributions of Ross Stable estimation. Panel A and B show that Ross Stable recovered probabilities are close to the market risk-neutral probabilities for both 20-day and 120-day horizons. This result is consistent with the empirical result of Jackwerth & Menner (2020). They also find the stochastic discount factor is stable and close to one.

Insert Figure (9)

The stable stochastic discount factor suggests the recovered probabilities, long-term risk-neutral probabilities as discussed byBorovička et al. (2016), may be close to risk-neutral probabilities empirically. Ross Basic estimation provides well-fitted risk-neutral probabilities in comparison to the market risk-neutral probabilities. The additional time-homogeneous transition matrix seems reasonable in Ross recovery. However, Ross Basic recovered probabilities and Ross Stable recovered probabilities present significant differences while they both have the same form of pricing kernel. It is not clear which approach provides recovered probabilities correctly reflecting the time-homogeneous pricing kernel. The rest of this paper tries to explain the differences between Ross Basic and Ross Stable mainly as a result of an omitted risk-free rate condition in Ross Basic.

Ross Stable estimation in the form of equation (14) is close to a linear system. But the right hand side is a power function of time discount factor δ. Jensen et al. (2019) propose a linear approximation $\delta^t \approx \alpha_t + \beta_t \delta$ around $\delta_0 = 0.97$ and Ross Stable becomes a linear system as

$$\begin{bmatrix} -\beta_1 & \pi_0(1,1) & ... & \pi_0(1,j) & ... & \pi_0(1,N) \\ -\beta_2 & \pi_0(2,1) & ... & \pi_0(2,j) & ... & \pi_0(2,N) \\ \vdots & ... & ... & ... & ... & \vdots \\ -\beta_T & \pi_0(T,1) & ... & \pi_0(T,j) & ... & \pi_0(T,N) \end{bmatrix} \times \begin{bmatrix} \delta \\ z_1/z_1 \\ \vdots \\ z_{k-1}/z_1 \\ z_k/z_1 \\ \vdots \\ z_N/z_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{T-1} \\ \alpha_T \end{bmatrix} \tag{16}$$

where $\alpha_t = -(t-1)\delta_0^t$ and $\beta_t = t\delta_0^{t-1}$, other variables are the same as equation (14). Jensen et al. (2019) present the closed-form solution for this system. However, the closed form solution z_k of this system may be negative. To ensure the non-negativity of z_k, an additional constraint $z_k \geq 0$ is required for this equation system. With this non-negativity constraint, there is no closed from solution. Instead, this closed form approach has to be estimated by a least squares estimation similarly as Ross Stable

$$\min_{z_k,\delta} \sum_{t=1}^{t=250} ((\sum_{k=1}^{N} \pi_0(t,k) z_k/z_1) - (\alpha_t + \beta_t \delta))^2 \quad s.t. \quad z_k \geq 0, \delta \geq 0 \tag{17}$$

this least squared problem is estimated by *lsqnonneg* function in MATLAB because it is already in the standard $C \cdot x - d$ form. This estimation is labelled as Ross Stable Linear.

Figure (10) compares the linear approximation and the original power function of δ^t. The linear approximation works well as $\alpha_t + \beta_t \delta$ is close to δ^t within one year horizon.

Insert Figure (10)

Figure (11) shows the Ross Stable Linear recovered probabilities and the market risk-neutral probabilities. Although the linear approximation only has small differences from the original δ^t. The recovered probabilities dramatically deviate from the Ross Stable recovered probabilities with many kinks. A small change in the discount factor results in a significant change in the pricing kernel. It suggests the importance of the discount factor in transition-independent recovery. Therefore, Ross Stable Linear approach has no advantage over the Ross Stable approach as δ^t is calculated in approximation. The empirical results of Ross Stable Linear are less reliable than Ross Stable.

Insert Figure (11)

Besides, Ross Basic implies a risk-free rate curve significantly different from the market risk-free rate curve. Ross Basic estimation fails to reflect the risk-free discount rates. The difference between Ross Stable Linear and Ross Stable brings a question of whether the difference between Ross Basic and Ross Stable is from Ross Basic implied risk-free rates.

4 Recovery with Interest Rates

In the previous section, Ross Basic and Ross Stable provide different recovered probabilities for the same spot state price surface. This section discusses the omitted constraint in Ross Basic and how flat interest rate curve results in a risk-neutral recovery.

4.1 Recovery and A Flat Interest Rate Term Structure

The analysis of interest rate term structure starts with a flat term structure. In an economy following the assumptions in Ross recovery or Ross Stable, no matter the shape of the spot state price surface, the recovered probabilities degenerate to risk-neutral probabilities when the term structure of risk-free rate is constant.

4.1.1 Ross Recovery with A Flat Interest Rate Curve

Based on Ross recovery, there is a recovered probability surface $\mathbf{P}$ from spot state price surface $\mathbf{\Pi}$ according to formula (9). A subgroup probability surface $\hat{\mathbf{P}}$ consists of $T = N$ consecutive horizons recovered probability from the original recovered probability surface $\mathbf{P}$. The probability surface $\hat{\mathbf{P}}$ is a squared probability matrix with the sum of each row as one. Assuming the probability surface $\hat{\mathbf{P}}$ is invertible

$$\begin{aligned} \hat{\mathbf{P}}\mathbf{e} &= \mathbf{e} \\ \mathbf{e} &= \hat{\mathbf{P}}^{-1}\mathbf{e} \end{aligned} \tag{18}$$

where $\mathbf{e}$ is a column unit vector.

In a Ross recovery economy, the one-period state interest rate vector R is estimated from the transition matrix shown as formula (10). The expected one-period short rate from any time t is $E(b_t) = P_t R$. Given a market short rate curve $\hat{B} = [b_1, b_2, ..., b_N]'$, and time-homogeneous one-period interest rate vector $R = [r_{t1}, r_{t2}, ..., r_T]'$

$$\hat{\mathbf{P}}R = \hat{B} \tag{19}$$

If the interest rate curve is flat, i.e. $\hat{B} = [b_1, b_2, ..., b_N]' = b\mathbf{e}$,

$$\begin{aligned} \hat{\mathbf{P}}R &= \hat{B} \\ R &= \hat{\mathbf{P}}^{-1}b\mathbf{e} \\ R &= b\mathbf{e} \end{aligned} \tag{20}$$

i.e. the one-period short rates on all trading dates are the same, Ross Recovery transition matrix implied interest rates are state independent and equal to the one-period short rate. According to the Ross recovery theorem: If the riskless rate is state independent, recovered probabilities are risk-neutral probabilities. The flat interest rate curve uniquely determines a stable pricing kernel in the Ross recovery theorem. Ross Recovery provides no additional information in comparison to risk-neutral estimation when the interest rate curve is flat.

This result is consistent with the critique of the Ross recovery theorem from Borovička et al. (2016).

Ross recovered probabilities is not physical probabilities but long-term risk-adjusted probability. When the interest rate is constant over time, the long-term risk-adjusted probabilities and the one-period risk-neutral probabilities are adjusted by the same interest rate.

4.1.2 Ross Stable with A Flat Interest Rate Curve

In Ross Stable/ Generalized recovery, the stochastic discount factor is transiton-independent. If the interest rate curve is flat across periods, it can be noticed that a unit vector and the risk-free discount factor are always one set of solution for equation system (14)

$$z_k = 1, \forall i \in N, and \quad \delta = exp(y_1) \tag{21}$$

where y_1 is one-period long rate.

This solution is independent of the shape of the spot state price surface. As long as the interest rate term structure is flat, risk-neutral estimation is always one solution of Ross Stable/ Generalized recovery. This result is similar to Ross recovery, indicating the assumption of transition-independent stochastic discount factors have an important impact on the recovery estimation. In practice, although the risk-free rate may be not flat, it is small and has relatively little variation during a short period. The numerical solution of (14) may be still close to formula (21) as shown in the empirical result of Ross Stable in previous section. But as Jensen et al. (2019) point out, in Generalized recovery/ Ross stable, the risk-neutral solution may be not unique under a constant interest rate curve.

4.2 Ross Recovery Estimation with An Interest Rate Condition

The market interest rate curve is usually not flat. E.g. the interest rate curve on January 4, 2019 increases with time in the short and medium terms but slightly decreases in the long term (see Figure (1)). This paper proposes a risk-free rate condition for Ross recovery estimation. The transition matrix in Ross recovery implies a short rate curve and a long rate curve as discussed in section (3.1.1). The expected one-period short rate under Ross recovery should equal the market one-period short rate. And the expected long rate for any horizon under Ross recovery should equal the market long rate.

4.2.1 Ross Short Rate Estimation

Ross Short Rate estimation has one more estimation term in comparison to Ross Basic estimation. Besides minimizing the errors between market state prices and fitted state prices, the transition matrix also tries to minimize the errors between the implied short rate and the market short rate.

$$\min_{a(i,j)} \sum_{j \in N} \sum_{t=0}^{230} (\pi_0(t+20,j) - \sum_{i \in N} \pi_0(t,i)a(i,j))^2 + \sum_{t=0}^{230} (b_t - \hat{b}_t)^2 \quad s.t. \quad a(i,j) \geq 0 \tag{22}$$

where b_t is the transition matrix implied short rate as formula (12), and $\hat{b}_t$ is the market short rate from the interest rate term structure, other variables are the same as the Ross Basic estimation.

However, this method has to find the Perron–Frobenius eigenvalue in each iteration during the optimization process. Ross Short Rate estimation is a nonlinear equation system. It is computation-intensive to find the local minimum. As a short rate curve can be transformed to a long rate curve directly, the following Ross Long Rate method provides an alternate approach to incorporating the interest rate curve without imposing a nonlinear constraint to Ross Basic.

4.2.2 Ross Long Rate Estimation

Ross Long Rate estimation is similar to Ross Short Rate estimation. Ross long rate estimation tries to minimize the errors between implied long rates and market long rates. It is more convenient to calculate the risk-free discount factor as the sum of state prices for any horizon t. Ross Long Rate estimation requires fitted risk-free discount factors equal market risk-free discount factors.

$$\min_{a(i,j)} \sum_{j \in N} \sum_{t=0}^{230} (\pi_0(t+20,j) - \sum_{i \in N} \pi_0(t,i)a(i,j))^2 + \omega \sum_{t=0}^{230} (\sum_{j \in N} (\pi_0(t+20,j) - \sum_{j \in N} \sum_{i \in N} \pi_0(t,i)a(i,j))^2 \quad s.t. \quad a(i,j) \geq 0 \tag{23}$$

where ω determines the relative importance of Ross Basic estimation and the interest rate condition, ω is set to a large value 100000 to ensure the interest rate condition is satisfied, all other variables are the same as Ross Basic. This optimization problem is estimated in a similar way as Ross Basic through *lsqnonneg* function in MATLAB[7].

Figure (12) shows the estimated state price transition matrix by Ross Long Rate approach. Like Ross Basic, because the spot state prices are close to zero in the deep tails. The transition matrix has some extreme values in the states corresponding to deep tails. Figure (12) Panel B shows the center part of the transition matrix. The state prices on the diagonal are higher than other states in the transition matrix.

Insert Figure (12)

Figure (13) panel A and B are fitted risk-neutral and recovered probabilities for 20-day and 120-day horizons. 20-day fitted risk-neutral probabilities are not significantly different from market risk-neutral probabilities. Overall, 120-day fitted risk-neutral probabilities are close to market risk-neutral probabilities.

[7] Least squared problem (23) can be written in a standard $C \cdot x - d$ form as Appendix A.2. *lsqnonneg* function in MATLAB provides a fast and reliable solution to the standard $C \cdot x - d$ problem.

However, the fitted risk-neutral probability distribution deviates from the market risk-neutral probability distribution around its mode. Unlike Ross Basic, the recovered probability distributions are close to the fitted risk-neutral probability distribution, especially for the 20-day horizon. Ross Long Rate approach provides similar results as the Ross Stable method, indicating a relatively stable pricing kernel. As the additional constraint of Ross Long Rate in comparison to Ross Basic is implied by the Ross recovery theorem, the strange shape of the Ross Basic recovered probability distribution is a result of the ill-conditioned estimation process. Ross Long Rate estimation complements the Ross recovery estimation by considering the implied long rate curve.

Insert Figure (13)

Figure (14) panel A shows one-period state interest rates. Like Ross Basic, Ross Long Rate estimation still results in states with large positive and large negative interest rates. But Ross Long Rate state interest rates are less volatile comparing to Ross Basic. Figure (14) panel B shows Ross Long Rate implies interest rates that are close to market interest rates. The Ross Long Rate implied interest rates from the one-period calculation are consistent with market interest rates, indicating an accurate estimation for the optimization problem (23). However, implied interest rates based on the multi-period calculation are not the same as the market interest rates. The Ross Long Rate estimation only minimizes the one-period estimation errors. Therefore, interest rates from the nonlinear multi-period calculation still deviate from the market interest rates.

Insert Figure (14)

The Ross recovery theorem is sensitive to the interest rate curve. Ross Basic estimation implies a volatile interest rate curve which results in a volatile pricing kernel. Ross Long Rate estimation implies a relatively stable interest rate curve which comes with a relatively stable pricing kernel. This empirical result complements the relationship between Ross recovery and the interest rate curve. A relatively stable interest rate curve results in a relatively stable pricing kernel. And a flat interest rate curve results in a constant pricing kernel.

5 Multi-Period Ross Recovery Estimation

In Ross Basic and other related estimations, the fitted state price at time t is represented as the market state price at time $t-1$ and the one-period transition matrix. However, based on the formula (6), fitted state prices at time t are also represented as market state prices at time t_0 and the power function of the transition matrix. This paper proposes a new approach as Ross Power estimation by minimizing the errors between the market state prices and the fitted state prices calculated with the power function of the transition matrix.

As Ross recovery heavily relies on the interest rate curve, Ross Long Rate estimation in the previous section provides a linear constraint of risk-free rate condition. However, implied long rates from the multi-period calculation are not consistent with market interest rates in Ross Long Rate estimation. This section proposes a multi-period Ross recovery estimation. Based on the formula (6), the spot state price surface is solely represented by the transition matrix. Following the same idea of overlapping spot state price surface in Ross Basic, spot state prices are specified as

$$\Pi_{i+20n} = \Pi_i \times \mathbf{A}^n, where \quad i = 0, 1, ..., 19, \quad n = 1, 2, ... \tag{24}$$

This multi-period representation is not a linear equation. It consists of n^{th} power function of the transition matrix for n^{th} period spot state prices. Besides, a penalty term ω on the error between the market risk-free discount factor and Ross implied risk-free discount factor for each horizon shows the relative importance of fitting the state prices and fitting the risk-free rate condition.

$$\begin{aligned} \min_{a(i,j)} \sum_{j \in N} \sum_{t=20}^{250} (\pi_0(t,j) - \hat{\pi}_0(t,j))^2 + \omega \sum_{t=20}^{250} (\sum \pi_0(t,j) - \sum \hat{\pi}_0(t,j))^2 \\ where\, \hat{\Pi} = [\hat{\pi}_0(t,j)] = \Pi_{t\ mod\ 20} \times \mathbf{A}^{floor(t/20)} \quad s.t. \quad a(i,j) \geq 0 \end{aligned} \tag{25}$$

where $\hat{\Pi}_t = [\hat{\pi}_0(t,j)]$ is the Ross Power method fitted spot state prices at time t calculated as formula (24).

If there is an accurate and unique Ross recovery transition matrix satisfying the spot state price surface, different approaches, including Ross Basic, Ross Root, Ross Long Rate, and Ross Power, will have the same solution for the Ross recovery theorem. However, the market spot state price surface usually implies no exact transition matrix for Ross recovery. Least squares estimations with different representations are not equivalent when there is no accurate solution. The least squares solution for Ross Basic or Ross Long Rate estimation is not necessarily the solution for Ross Power estimation. Ross Power estimation has the advantage of correctly representing the long-term state prices and interest rate as a result of the n^{th} power function of the transition matrix. However, the estimation speed of Ross Power also suffers from the nonlinear representation.

In previous linear least squares representations of Ross recovery estimation, there are 100 states which result in a transition matrix with $100^2 = 10000$ unknown parameters. To reduce the computation intensity of Ross Power, this section uses a simplified spot state price surface with 20 states from moneyness 0 to moneyness 2 with an interval as moneyness 0.1. Thus, the number of unknown parameters in the nonlinear least squares problem (25) decreases from 10000 to 400 which makes the computation feasible. Also, the nonlinear high degree least squares problem may have many local minimum values. To compare with Ross

Long Rate, the Ross Power estimation uses the result of Ross Long Rate with the same sample data as the starting guess in the optimization process.

Figure (15) shows the estimated state price transition matrix by Ross Power approach. Like other Ross recovery estimations, the transition matrix has some extreme values in the states corresponding to the deep tails. And the state prices at the main diagonal of the transition matrix are much higher than other state prices in the transition matrix.

Insert Figure (15)

Figure (16) panel A and B are the fitted risk-neutral and recovered probabilities for 20-day and 120-day horizons. The Ross Power fitted risk-neutral probability distribution has more deviations from the market risk-neutral probability distribution as a result of more strict constraints from the power function of the transition matrix. Similarly to Ross Long Rate estimation, Ross Power recovered probability is close to Ross fitted risk-neutral probability.

Insert Figure (16)

Figure (17) panel A shows one-period state interest rates. Ross Power implies relatively stable state interest rates around at-the-money states. Only the states corresponding to the deep left tail of market risk-neutral distribution implies extreme state interest rates. Besides, the Ross Power implied state interest rate is close to the market interest rate. Figure (17) panel B shows Ross Power estimation implies an interest rate curve that is almost the same as the market interest rate. The nonlinear Ross Power estimation correctly represents the long-term risk-free rate in comparison to other Ross recovery approaches.

Insert Figure (17)

Overall, Ross Power estimation implies an interest rate curve consistent with the market interest rate curve and has a recovered probability distribution only slightly deviating from the risk-neutral probability distribution. This is consistent with the idea that as Ross implied interest rate term structure becomes flat, Ross recovered probabilities approach to risk-neutral probabilities. The empirical analysis shows when a market interest rate curve is not flat but short-term upward-sloping, Ross recovered probabilities are still close to risk-neutral probabilities. However, due to computation difficulty in the high degree large nonlinear least squares estimation, further research with the computation process may help the empirical application of Ross Power estimation.

6 Ross Recovery Implied Information

This section compares the implied information from different approaches of Ross recovery. Because of the computation limitation in the Ross Power approach, this section estimates different Ross recovery approaches using the relatively sparse spot state price surface with 20 states from moneyness 0 to moneyness 2 with an interval as moneyness 0.1.

The accuracy of the transition matrix is estimated as the Sum of Squared Errors (SSE) between the fitted state prices and the market state prices from 20 days to 250 days. Estimation accuracies of Ross Stable and Ross Stable Linear are estimated as the Sum of Squared Errors between the sum of recovered probability and 100% as there is no transition matrix. Table (1) shows the estimation accuracies of different Ross recovery approaches. Overall, there are only small errors in all Ross recovery approaches. The estimated one-period transition matrix provides a reliable representation for the whole spot state price surface. The one-period transition matrix provides new forecasting about the long-term risk-neutral probability distribution that is not directly observable in the market. SSE of Ross Basic is the smallest among all transition matrix dependent Ross recovery approaches as Ross Basic has the most feasible transition matrix. The largest SSE of Ross Root indicates the implicit requirement for a nonnegative root for long-term transition matrix is a strong constraint for Ross recovery estimation. However, the optimal length of the one-period transition time still requires further research. Ross Long Rate and Ross Power estimations have greater SSE than Ross Basic because of an additional risk-free rate condition. Applications of Ross recovery have to determine the relative importance between fitting the overall spot state price surface and guaranteeing the term structure of risk-free rates. If the risk-free rate term structure is ignored, a linear estimation of Ross recovery is Ross Basic with small SSE. If fitting the risk-free rate term structure is the priority, such as Ross Long Rate estimation, relatively large SSE comes with the risk-free condition. Besides, the SSE of Ross Stable and Ross Stable Linear is extremely small. Ross Stable only uses the time-homogeneous stochastic discount factor assumption in Ross recovery without assuming a time-homogeneous transition matrix. Therefore Ross Stable has the most feasible pricing kernel with a small SSE.

Insert table (1)

The stochastic discount factors show the pricing kernel between the risk-neutral measure and Ross recovery measure. Figure (18) compares the 20-day stochastic discount factors of all approaches to Ross recovery theorem on January 4, 2019 with 20-state spot state price surface. Extreme recovered probabilities and volatile shapes of stochastic discount factors in Ross Basic and Ross Root are probably due to the ill-conditioned estimation process.

Unlike the result of the 100-state state price surface (see the difference of fitted risk-neutral distribution and recovered probability distribution as Figure (11)), stochastic discount factors of Ross Stable Linear are similar to that of Ross Stable with the 20-state state price surface. Fitting the transition matrix in a dense spot state price surface is more likely to result in a relatively volatile pricing kernel. Ross Long Rate approach with a 20-state state price surface also implies a more stable pricing kernel to Ross Long Rate with a 100-state state price surface.

Ross Stable, Ross Stable Linear, Ross Long Rate, and Ross Power all have relatively stable pricing kernels, especially around at-the-money. The relatively stable stochastic discount factor suggests similar information between the risk-neutral measure and the Ross recovery measure. Because Ross Long Rate and Ross Power require a time-homogeneous transition matrix in comparison to Ross Stable. Ross Long Rate and Ross Power have relatively more volatile pricing kernels.

Insert Figure (18)

Some Markov Chain processes converge to stationary distributions in the long term. This paper selects a long-term horizon, 10 years, to present the convergence of the Markov Chain process in Ross recovery. Because the sum of spot state prices reflects the risk-free discount rate that is horizon dependent. This section focuses on the long-term risk-neutral and recovered distributions as shown in Figure (19). The time-homogeneous transition matrix estimated from a one-year spot state price surface implies approximately bimodal risk-neutral distributions in the long term with an upside movement and a downside movement in the underlying asset price. Market option contracts on January 4, 2019 include expiries only within a horizon of 258 days. The transition matrix provides a novel method for long-term expectations. Although Ross recovered probabilities are empirically close to the risk-neutral probabilities, the reasons for the implied long-term bimodal risk-neutral distribution are unclear. Similar to risk-neutral distributions, recovered probability distributions also converge to their long-term stationary distributions. Consistent to figure (18) of the stochastic discount factors, the stationary recovered probabilities of Ross Basic and Ross Root indicate strange shapes while the stationary recovered probabilities of Ross Long Rate and Ross Power are close to the fitted risk-neutral probability distributions.

Insert Figure (19)

Table (2) quantifies the differences between fitted risk-neutral probabilities and recovered probabilities of different Ross approaches with their moments. For one-period horizon, moments of fitted risk-neutral distributions are close to moments of market risk-neutral distributions, indicating again a relatively accurate fitting of the transition matrix with the market state price surface. The first four moments of recovered probability distributions from Ross Long Rate, Ross Power, Ross Stable, and Ross Stable Linear are similar to their corresponding fitted risk-neutral moments while recovered moments of Ross Basic and Ross Root are largely different from their corresponding fitted risk-neutral moments. This is consistent with the shapes of their stochastic discount factors in Figure (18). Table (2) panel B confirms in the long term, moments of both recovered probabilities and fitted risk-neutral probabilities from Ross Long Rate and Ross Power are close to each other while moments of recovered probabilities from Ross Basic and Ross Root are different from other approaches.

Insert Table (2)

7 Conclusion

This paper firstly proposes the importance of the term structure of risk-free rate in Ross recovery estimation. When the term structure of risk-free rate is flat, Ross recovery or Generalized recovery is always the same as a risk-neutral estimation. When market risk-free rates are not constant across different horizons, a risk-free condition is necessary for Ross recovery. Using market risk-free rates on January 4, 2019 as an example, empirical Ross recovered probabilities are still close to risk-neutral probabilities. Previous literature omits this risk-free condition in Ross recovery. By incorporating the risk-free rate condition into Ross recovery, this paper explains the difference between the original Ross recovery estimation and Generalized recovery/ Ross Stable estimations.

Besides, this paper presents the length of one-period transition time and the specification of least squares estimation affect recovered probabilities. A short one-period transition time in Ross recovery implies the requirement for a nonnegative root of long-term transition matrix which is a strong extra constraint in Ross recovery estimation. Selecting a longer one-period transition time would fit better with the market spot state price surface when all else equal.

Linear least squares representation of Ross recovery is a fast estimation. However, when there is no unique and exact transition matrix satisfying the spot state price surface, the linear representation is not accurate in representing the term structure of Ross recovery implied risk-free rate. A nonlinear representation of Ross recovery estimation solves this problem but significantly increasing the computation time. Further research about the optimization calculation of the nonlinear representation of Ross recovery may provide a more accurate estimation of Ross recovery estimation.

Appendix A Least Squares Estimation for Ross recovery

This section explicitly provides the standard $C \cdot x - d$ form linear equation system for Ross recovery estimation. In a standard $C \cdot x - d$ form linear equation system, x is a column vector with n unknown variables, C is a matrix with n columns, d is a column vector with the same length as the rows of matrix C. The MATLAB function *lsqnonneg* provides a fast and reliable solution to a Standard $C \cdot x - d$ form linear equation system.

A.1 Least Squares Estimation for Ross Basic

Given the market spot state price surfaces $\mathbf{\Pi}_a = [\Pi'_{a1}, ..., \Pi'_{ai}, ..., \Pi'_{aT}]'$, where Π_{ai} is the row vector in $\mathbf{\Pi}_a$, $\mathbf{\Pi}_b = [\Pi'_{b1}, ..., \Pi'_{bi}, ..., \Pi'_{bT}]'$, where Π_{bi} is the row vector in $\mathbf{\Pi}_b$, and the state price transition matrix $\mathbf{A} = [A_1, A_2, ..., A_N]$, where A_j is the column vector in the transition matrix. A row vector with all zero entries $\mathbf{0}_N = [0, ..., 0]_N$ has the same length as the number of states N in transition matrix A. The original matrix equation $\mathbf{\Pi}_b = \mathbf{\Pi}_a \times \mathbf{A}$ is equivalent to a new equation system as

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_i \\ \vdots \\ C_T \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix} = \begin{bmatrix} \Pi'_{b1} \\ \Pi'_{b2} \\ \vdots \\ \Pi'_{bi} \\ \vdots \\ \Pi'_{bT} \end{bmatrix}, where \quad C_i = \begin{bmatrix} \Pi_{ai} & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \mathbf{0}_N & \Pi_{ai} & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \Pi_{ai} & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \Pi_{ai} \end{bmatrix} \tag{A.1}$$

such that the new equation system follows the standard $C \cdot x - d$ form.

Similarly, the Ross Root estimation also has a standard $C \cdot x - d$ form by replacing the transition period in Ross Basic to a shorter period.

A.2 Least Squares Estimation for Ross Long Rate

Ross Long Rate estimation has an additional risk-free rate condition in comparison to Ross Basic estimation. The sum of fitted state prices equals the sum of market state prices for any horizon. With a unit row vector $\mathbf{e} = [1, ..., 1]_N$, the risk-free rate condition is

$$\begin{bmatrix} \mathbf{e}C_1 \\ \mathbf{e}C_2 \\ \vdots \\ \mathbf{e}C_i \\ \vdots \\ \mathbf{e}C_T \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix} = \begin{bmatrix} \sum \Pi_{b1} \\ \sum \Pi_{b2} \\ \vdots \\ \sum \Pi_{bi} \\ \vdots \\ \sum \Pi_{bT} \end{bmatrix}, where \quad C_i = \begin{bmatrix} \Pi_{ai} & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \mathbf{0}_N & \Pi_{ai} & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \Pi_{ai} & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \Pi_{ai} \end{bmatrix} \tag{A.2}$$

This risk free rate condition is applied to Ross Basic estimation,

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_i \\ \vdots \\ C_T \\ \omega\mathbf{e}C_1 \\ \omega\mathbf{e}C_2 \\ \vdots \\ \omega\mathbf{e}C_i \\ \vdots \\ \omega\mathbf{e}C_T \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix} = \begin{bmatrix} \Pi'_{b1} \\ \Pi'_{b2} \\ \vdots \\ \Pi'_{bi} \\ \vdots \\ \Pi'_{bT} \\ \omega\sum \Pi_{b1} \\ \omega\sum \Pi_{b2} \\ \vdots \\ \omega\sum \Pi_{bi} \\ \vdots \\ \omega\sum \Pi_{bT} \end{bmatrix} whereC_i = \begin{bmatrix} \Pi_{ai} & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \mathbf{0}_N & \Pi_{ai} & ... & \mathbf{0}_N & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \Pi_{ai} & ... & \mathbf{0}_N \\ \vdots & ... & ... & ... & ... & \vdots \\ \mathbf{0}_N & \mathbf{0}_N & ... & \mathbf{0}_N & ... & \Pi_{ai} \end{bmatrix} \tag{A.3}$$

where the weight parameter ω (ω is 100000 in this paper to guarantee the risk-free rate condition is satisfied) determines the relative importance between Ross Basic estimation and the risk-free rate condition. Therefore, the Ross Long Rate estimation as (A.3) is a standard from $C \cdot x - d$ least square problem.

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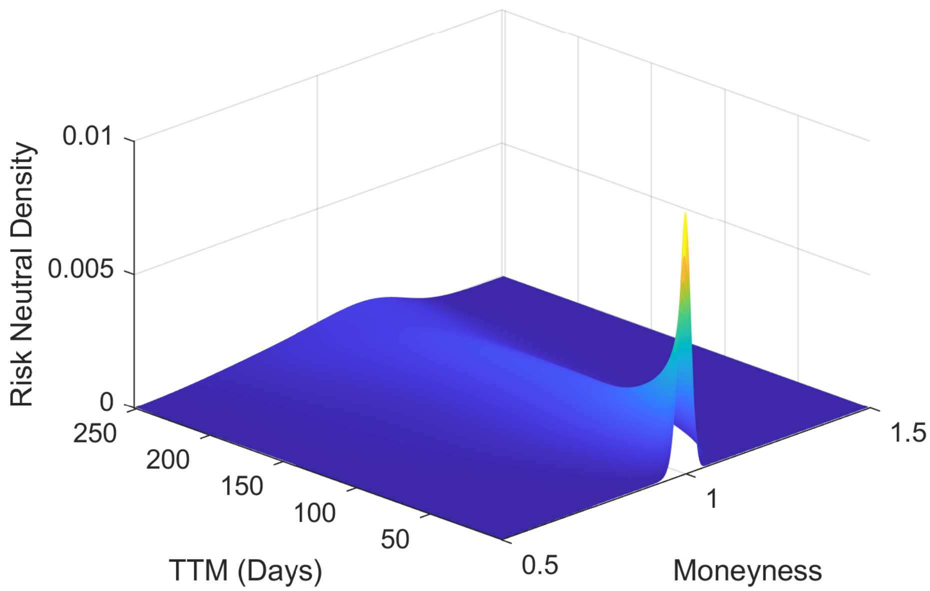
Table 1: Estimation Accuracy.
This table shows the Sum of Squared Errors (SSE) of different Ross recovery approaches with the SPXW option data on January 4, 2019. SSE of Ross Basic, Ross Root, Ross Long Rate, and Ross Power shows the difference between Ross fitted state price surface and the market state price surface. SSE of Ross Stable and Ross Stable Linear show the difference between the sum of the recovered probability and 100%.

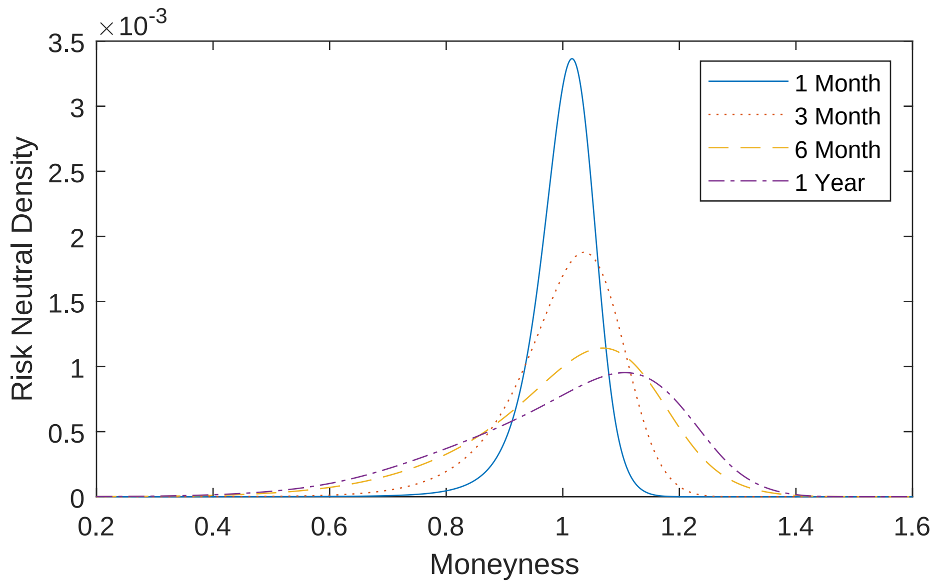
	SSE
Ross Basic	2.64e-02
Ross Root	6.74e-02
Ross Long Rate	3.14e-02
Ross Power	3.14e-02
Ross Stable	2.40e-30
Ross Stable Linear	2.40e-30

Table 2: Moments.
This table shows the moments of fitted and recovered distributions from different Ross recovery estimations with the SPXW option data on January 4, 2019. In panel A, mean and standard deviation are annualized.

Panel A: One Period					
		Mean (%)	Std (%)	Skewness	Kurtosis
Market Risk-neutral probability		0.85	20.36	-0.95	4.37
Ross Basic	Fitted	1.00	20.16	-0.87	4.27
	Recovered	-1.53	27.14	-0.64	3.80
Ross Root	Fitted	0.92	19.87	-0.82	4.18
	Recovered	-19.32	31.55	-0.08	2.92
Ross Long Rate	Fitted	0.87	20.37	-0.96	4.40
	Recovered	0.85	20.39	-0.96	4.41
Ross Power	Fitted	1.17	19.97	-0.85	4.30
	Recovered	1.06	20.02	-0.84	4.30
Ross Stable		1.17	19.97	-0.85	4.30
Ross Stable Linear		1.06	20.02	-0.84	4.30
Panel B: Long Term Stationary					
		Mean (%)	Std (%)	Skewness	Kurtosis
Ross Basic	Fitted	1.23	22.05	-0.32	2.01
	Recovered	9.14	26.00	-0.85	2.53
Ross Root	Fitted	1.74	19.83	-0.48	2.78
	Recovered	0.83	31.03	-0.49	2.15
Ross Long Rate	Fitted	0.93	21.08	-0.37	2.12
	Recovered	0.83	21.15	-0.37	2.14
Ross Power	Fitted	0.93	21.08	-0.37	2.12
	Recovered	0.83	21.15	-0.37	2.14



Panel A: RND Surface.



Panel B: Term Structure of RNDs

Figure 1: RND surface of SPXW options on January 4, 2019. Panel A shows the risk-neutral distribution surface. Panel B selects four RNDs from the surface.

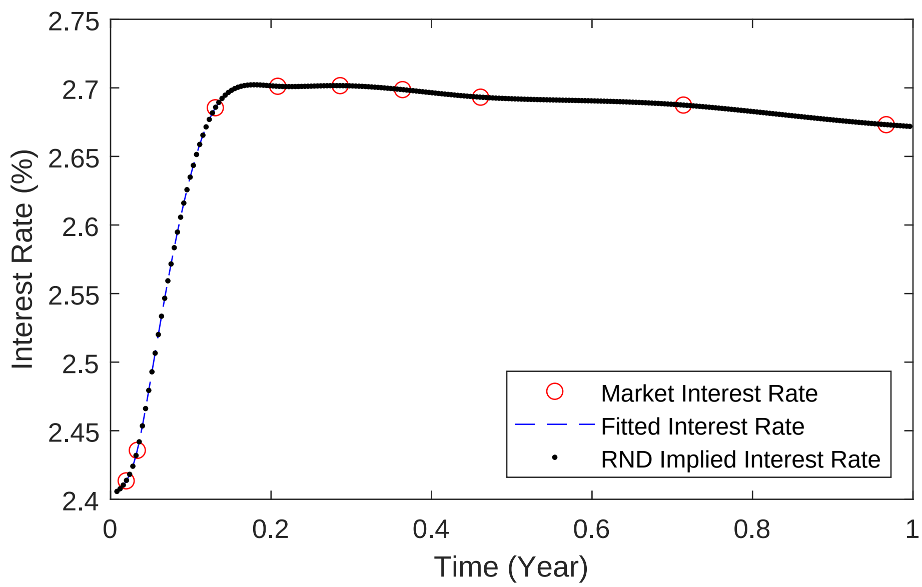
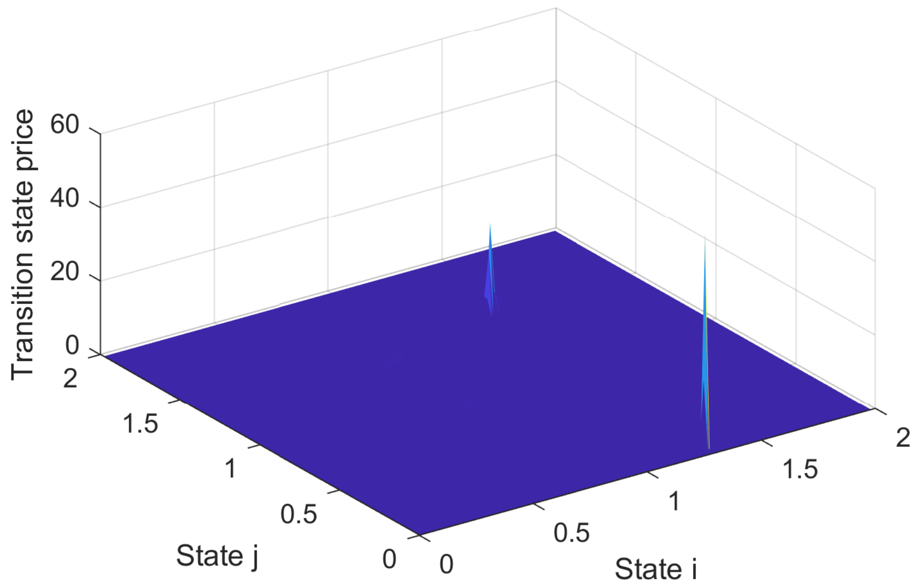
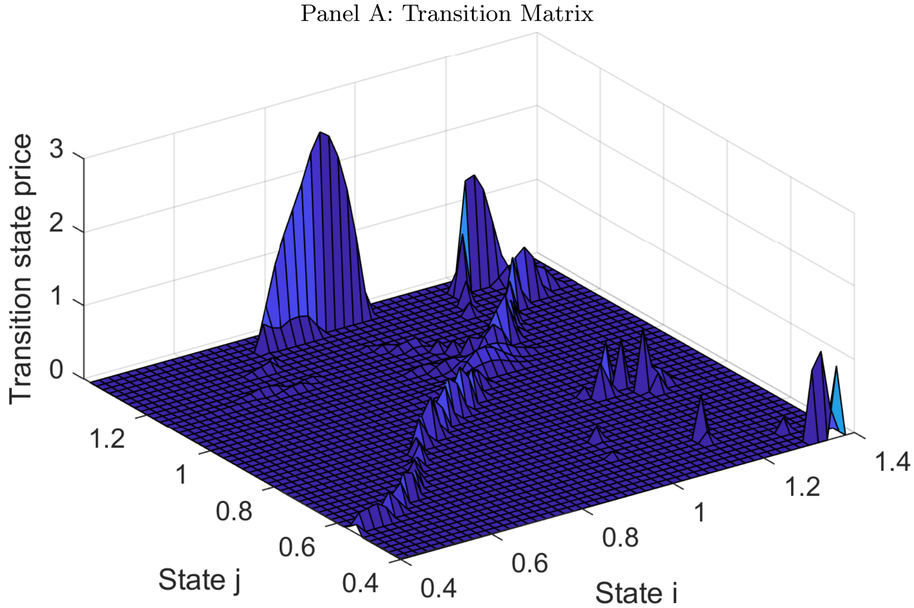


Figure 2: A comparison between RND implied interest rates and the market interest rates. Market interest rates are based on the US zero-coupon yield. Fitted interest rates are interpolated interest rates. RND implied interest rates are the expected returns of the RND plus the corresponding dividend yields for each horizon. Sample date is on January 4, 2019.

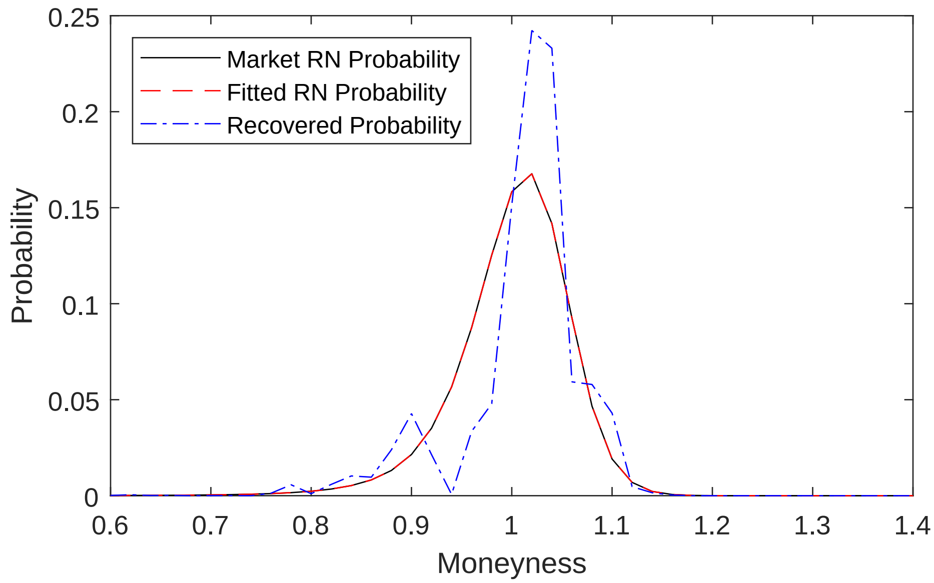


Panel A: Transition Matrix

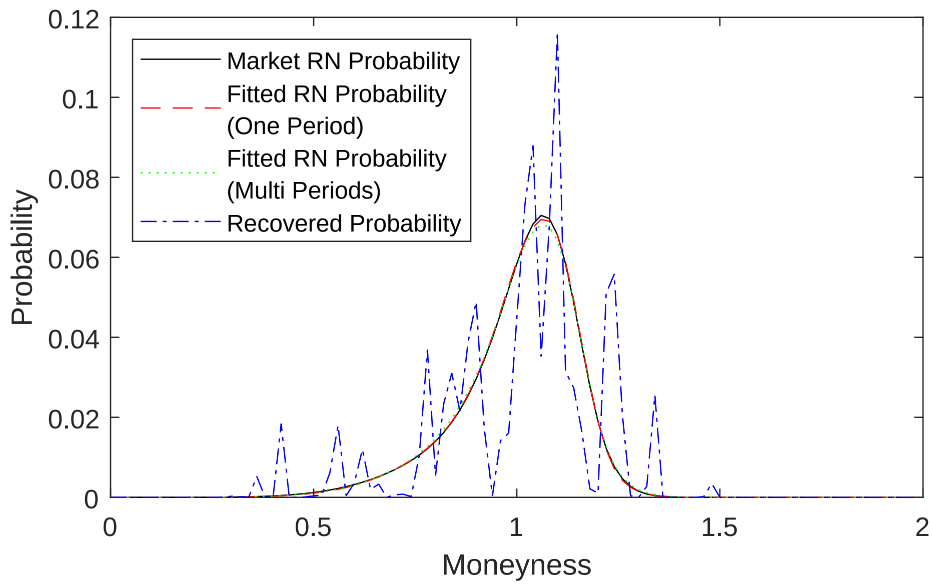


Panel B: Transition Matrix Center

Figure 3: Ross Basic Transition Matrix. This figure shows the one-period state price transition matrix based on Ross Basic estimation of SPXW options on January 4, 2019. Panel B focused on the states with moneyness from 0.4 to 1.4. State i is the state at time t. State j is the state at time $t + 1$.

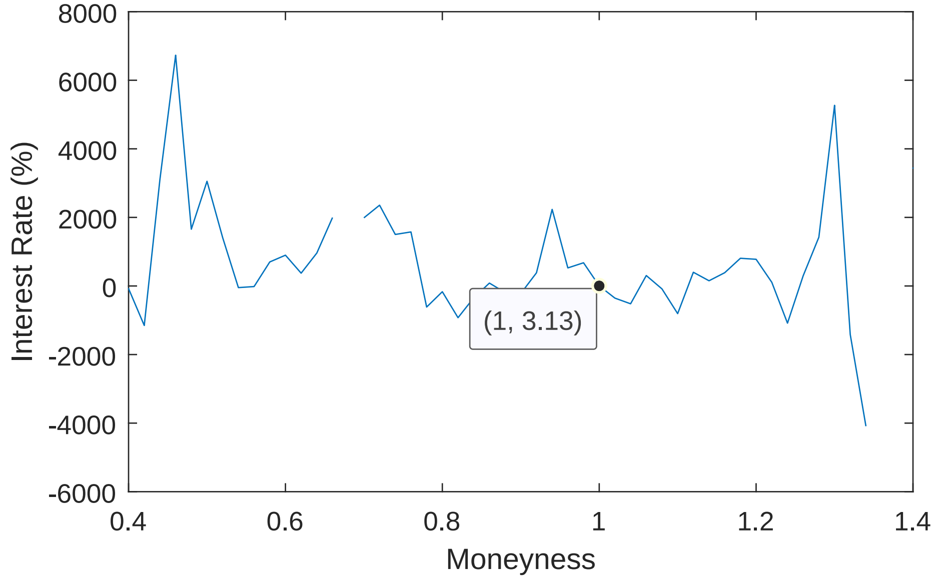


Panel A: 20-day

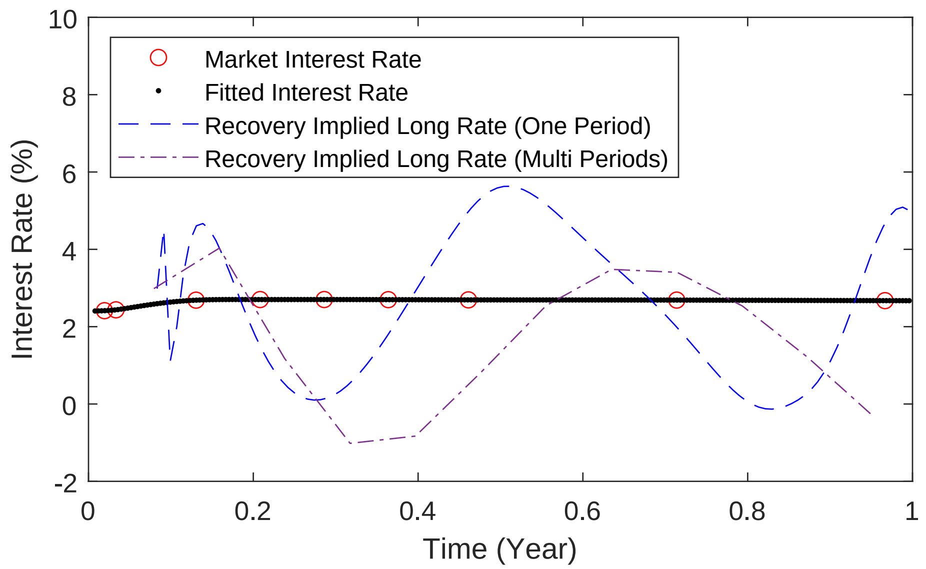


Panel B: 120-day

Figure 4: Recovered Ross Basic Probability Distributions. This figure shows fitted risk-neutral (RN) probabilities and recovered probabilities based on Ross Basic estimation of SPXW options on January 4, 2019. Fitted RN Probabilities (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Fitted RN Probabilities (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.

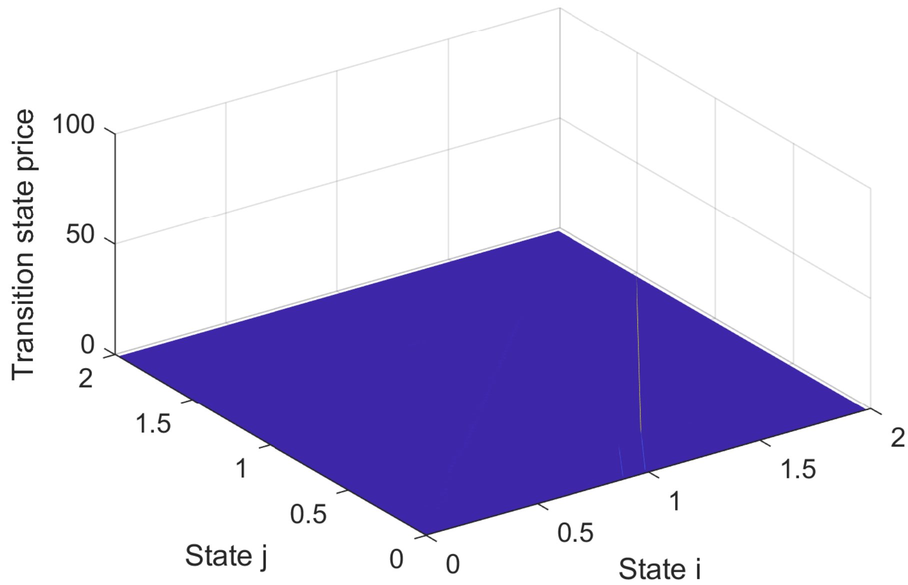


Panel A: State Interest Rate

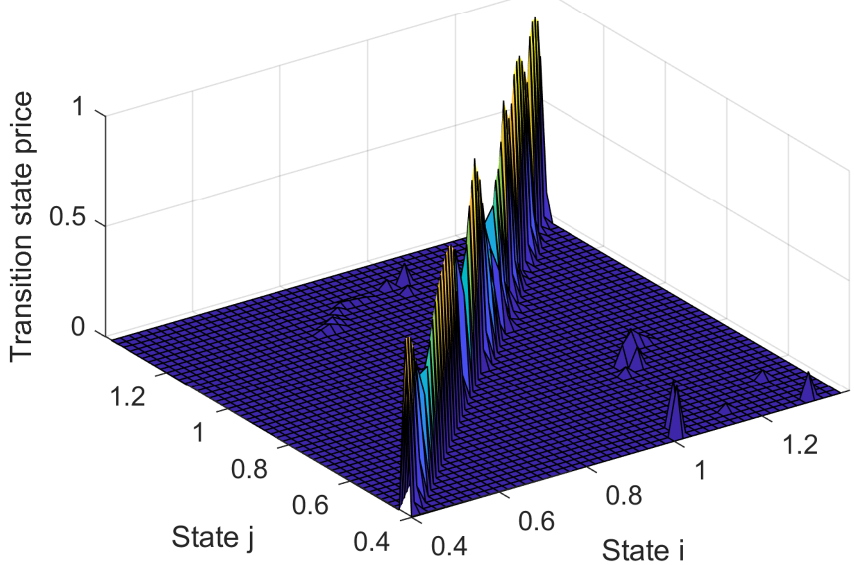


Panel B: Long Rate Curve

Figure 5: Ross Basic Implied Interest Rates. This figure shows the implied interest rates based on Ross Basic estimation of SPXW options on January 4, 2019. Panel A shows the one-period state interest rates. In Panel B, Recovery implied long rates (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Recovery implied long rates (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.

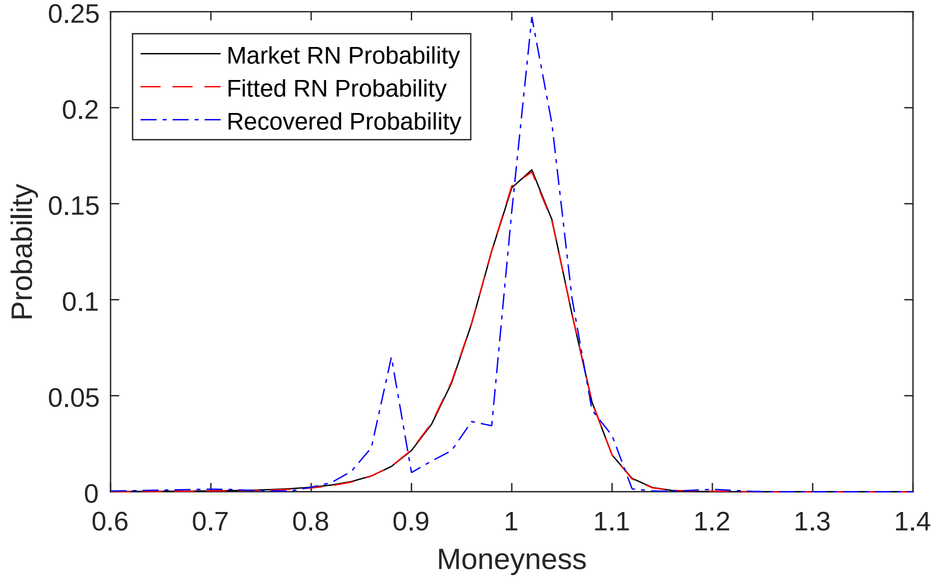


Panel A: Transition Matrix

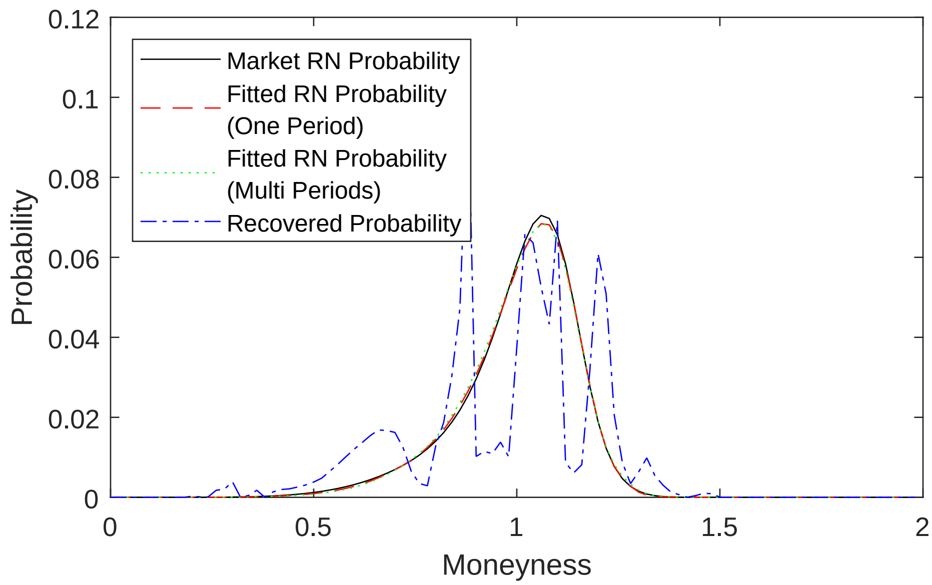


Panel B: Transition Matrix Center

Figure 6: Ross Root Transition Matrix. This figure shows the one-period state price transition matrix based on Ross Basic estimation of SPXW options on January 4, 2019. Panel B focused on the states with moneyness from 0.4 to 1.4. State i is the state at time t. State j is the state at time $t+1$.

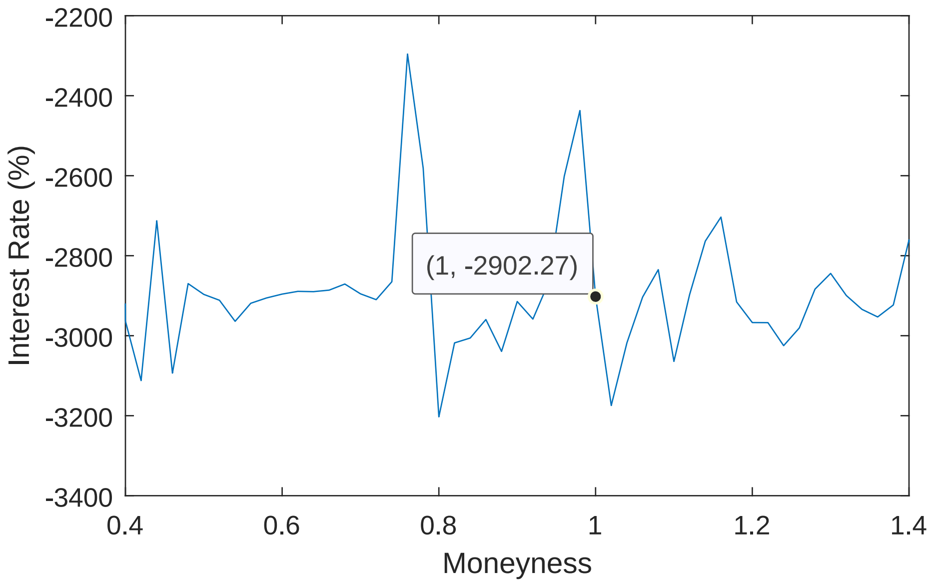


Panel A: 20-day

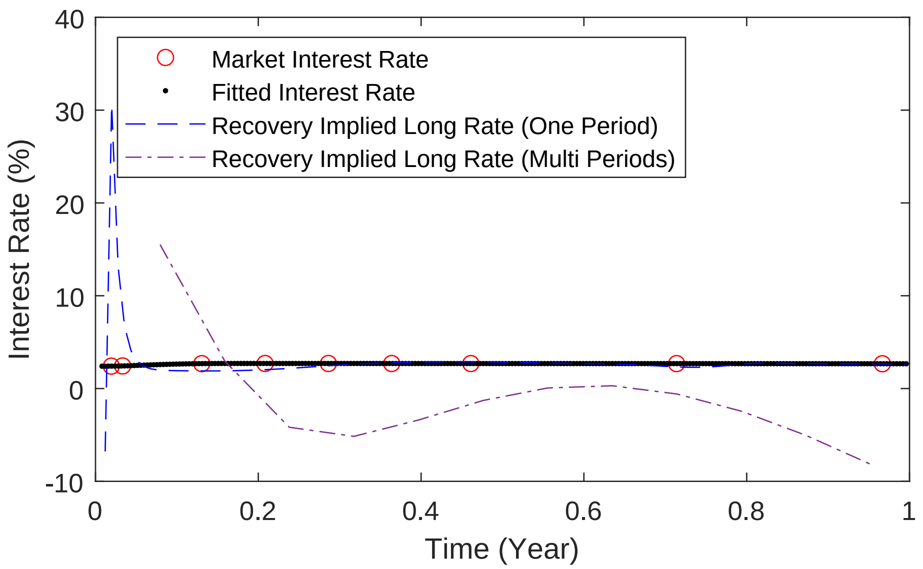


Panel B: 120-day

Figure 7: Recovered Ross Root Probability Distributions. This figure shows fitted risk-neutral (RN) probabilities and recovered probabilities based on Ross Root estimation of SPXW options on January 4, 2019. Fitted RN Probabilities (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Fitted RN Probabilities (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.

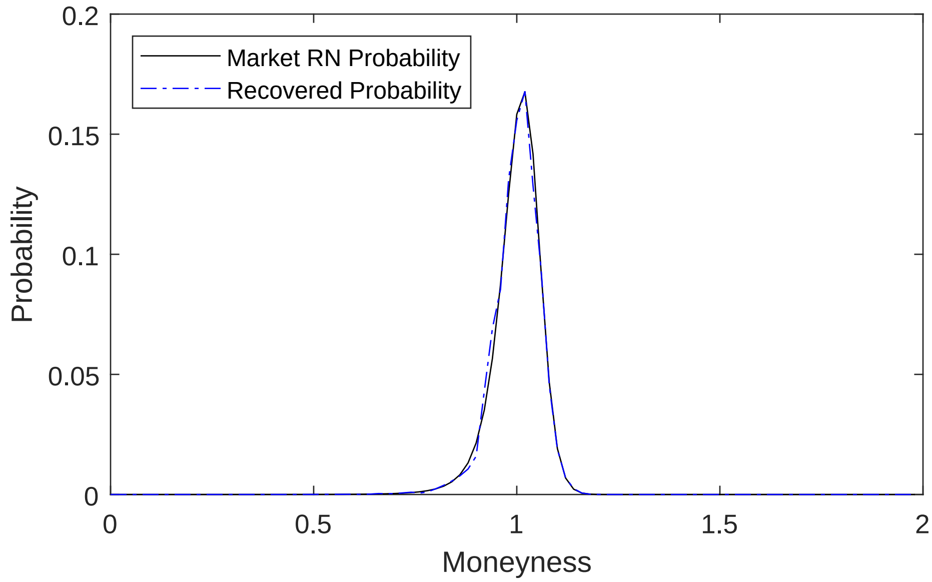


Panel A: State Interest Rate

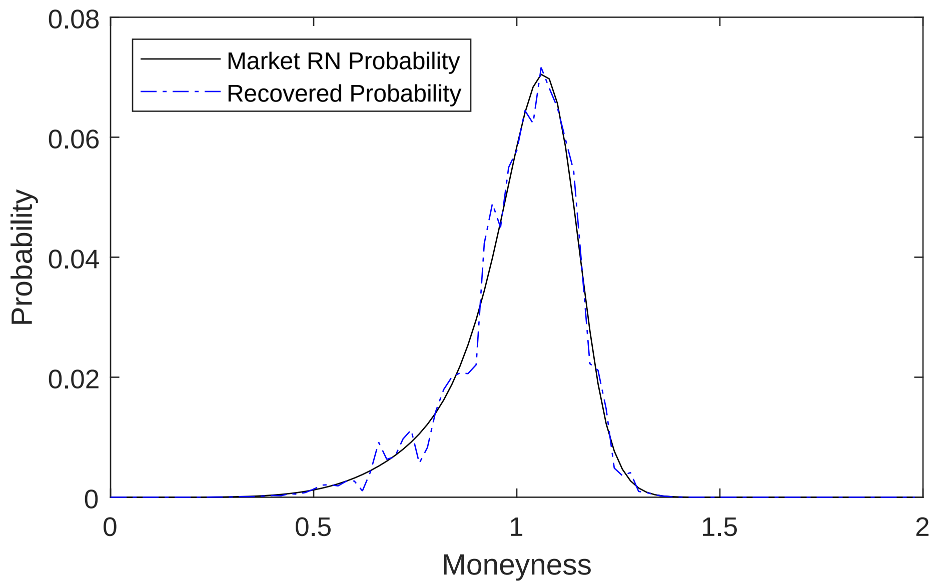


Panel B: Long Rate Curve

Figure 8: Ross Root Implied Interest Rates. This figure shows the implied interest rates based on Ross Root estimation of SPXW options on January 4, 2019. Panel A shows the one-period state interest rates. In Panel B, Recovery implied long rates (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Recovery implied long rates (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.



Panel A: 20-day



Panel B: 120-day

Figure 9: Recovered Ross Stable Probability Distributions. This figure shows market risk-neutral (RN) probabilities and recovered probabilities based on Ross Stable estimation of SPXW options on January 4, 2019.

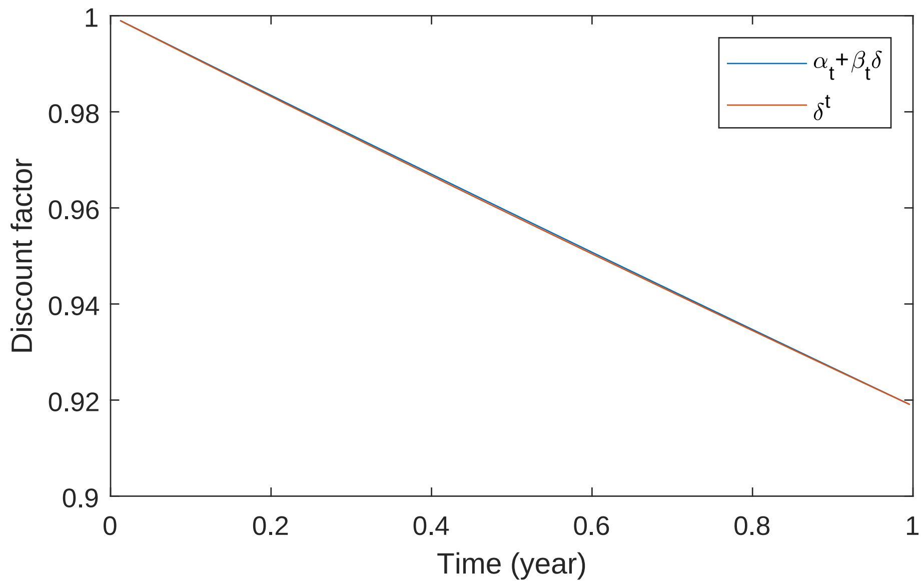
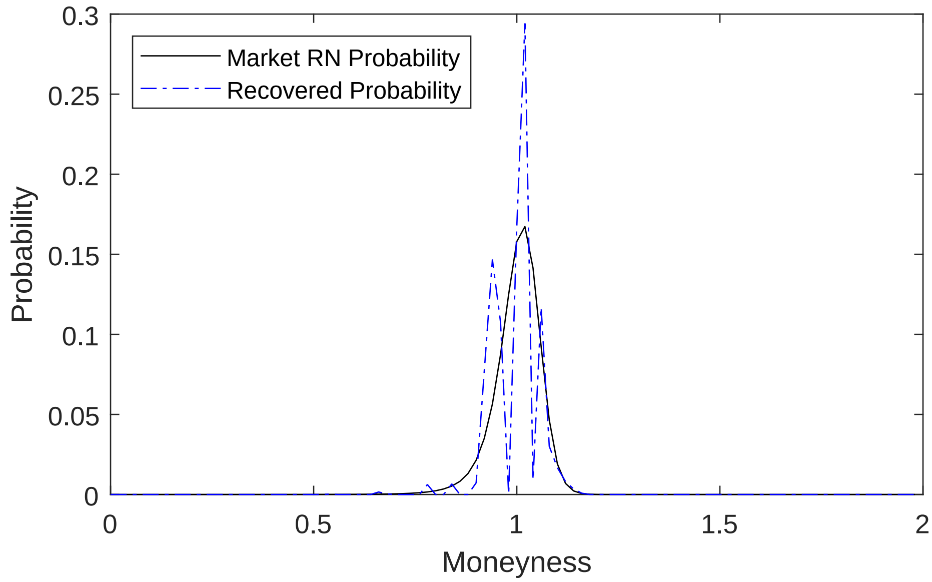
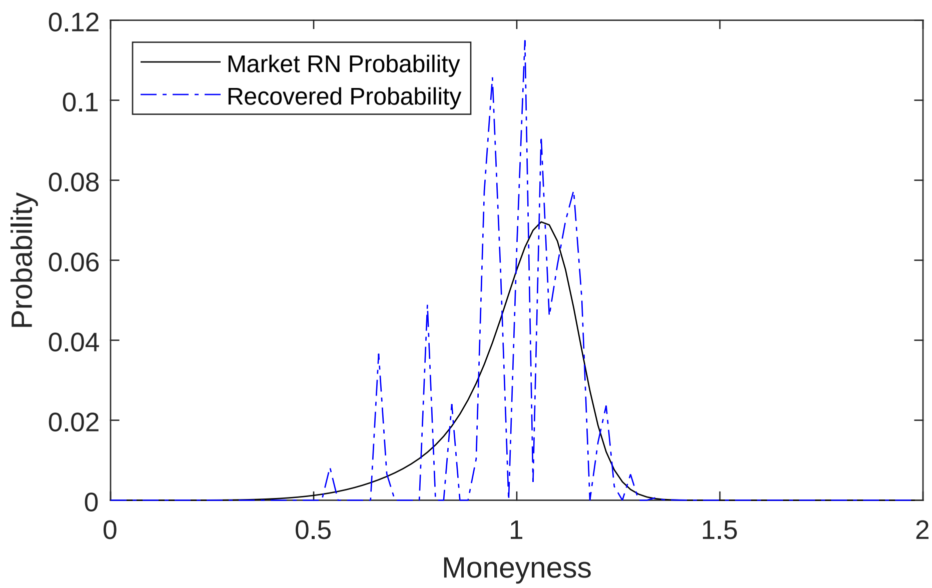


Figure 10: The Linear Approximation of Discount Factor. This figure shows the linear approximation of the discount factor as $\alpha_t + \beta_t\delta$ and the original discount factor δ^t based on Ross Stable Linear estimation of SPXW options on January 4, 2019.

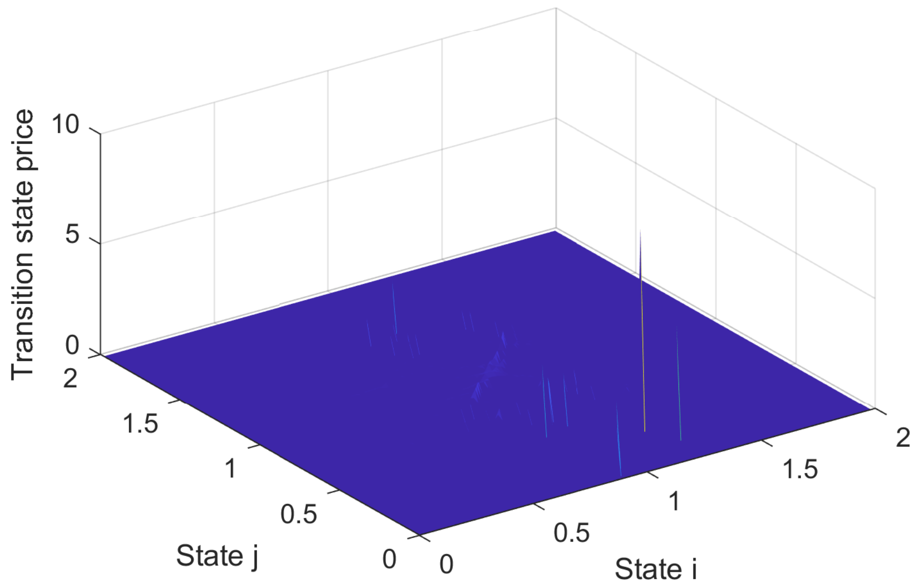


Panel A: 20-day

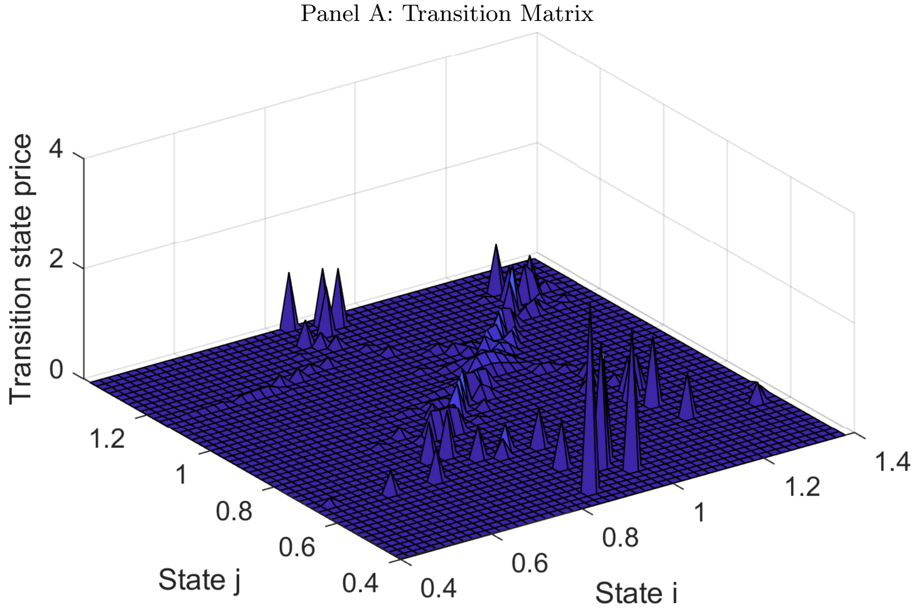


Panel B: 120-day

Figure 11: Recovered Ross Stable Linear Probability Distributions. This figure shows market risk-neutral (RN) probabilities and recovered probabilities based on Ross Stable Linear estimation of SPXW options on January 4, 2019.

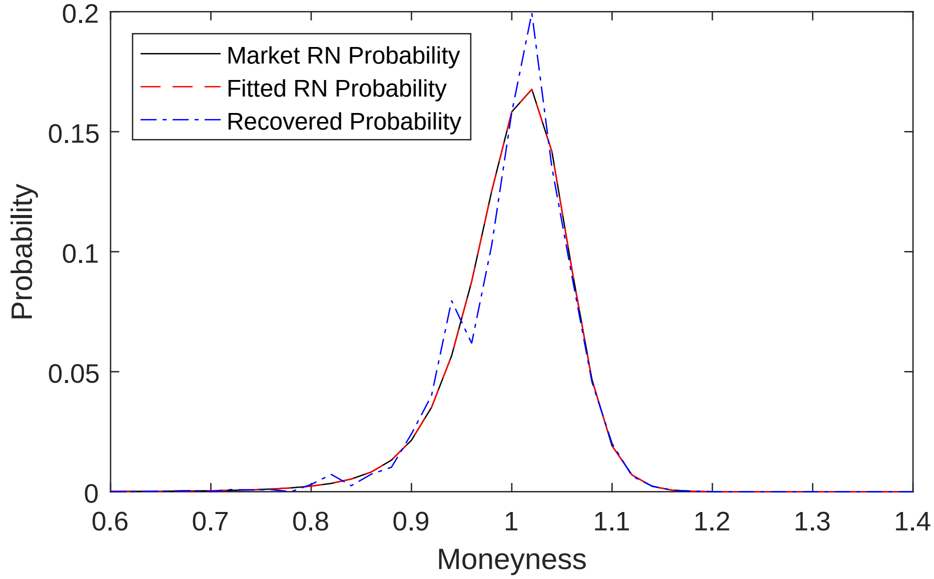


Panel A: Transition Matrix

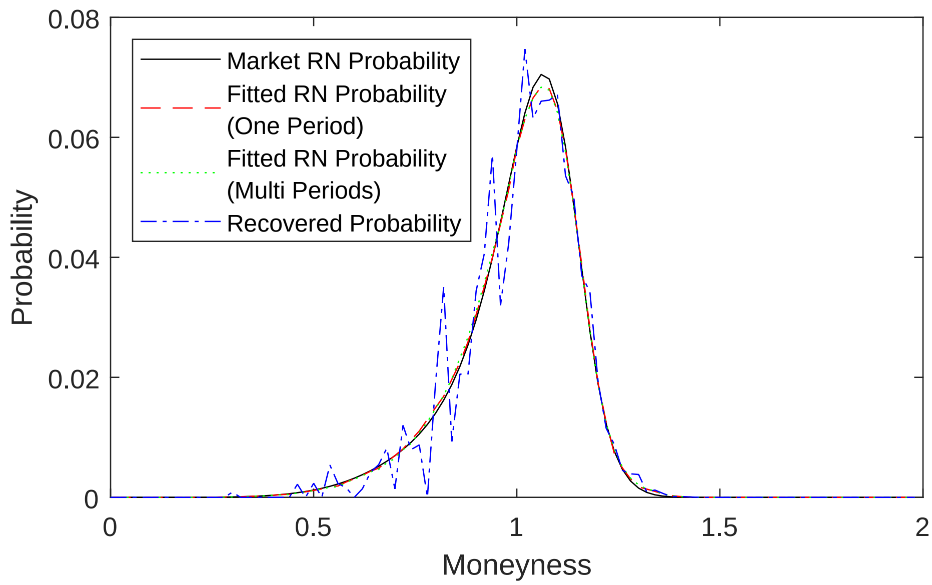


Panel B: Transition Matrix Center

Figure 12: Ross Long Rate Transition Matrix. This figure shows the one-period state price transition matrix based on Ross Long Rate estimation of SPXW options on January 4, 2019. Panel B focused on the states with moneyness from 0.4 to 1.4. State i is the state at time t. State j is the state at time $t+1$.

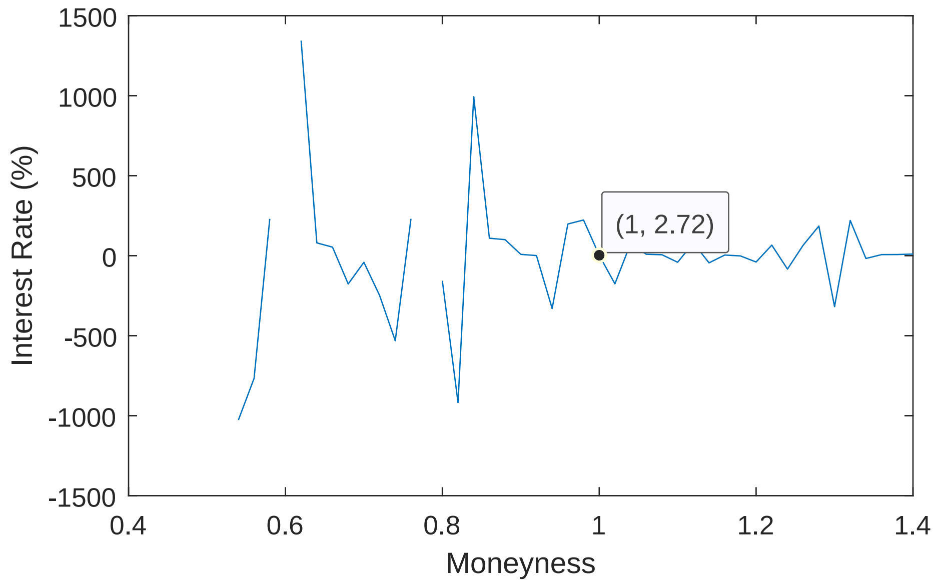


Panel A: 20-day

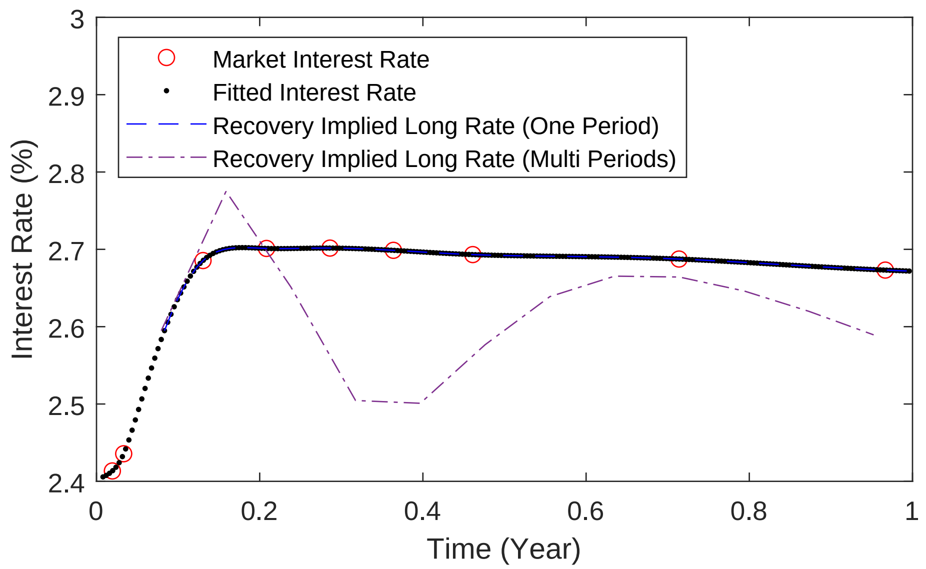


Panel B: 120-day

Figure 13: Recovered Ross Long Rate Probability Distributions. This figure shows fitted risk-neutral (RN) probabilities and recovered probabilities based on Ross Long Rate estimation of SPXW options on January 4, 2019. Fitted RN Probabilities (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Fitted RN Probabilities (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.



Panel A: State Interest Rate.



Panel B: Long Rate Curve

Figure 14: Ross Long Rate Implied Interest Rates. This figure shows the implied interest rates based on Ross Long Rate estimation of SPXW options on January 4, 2019. Panel A shows the one-period state interest rates. In Panel B, Recovery implied long rates (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Recovery implied long rates (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.

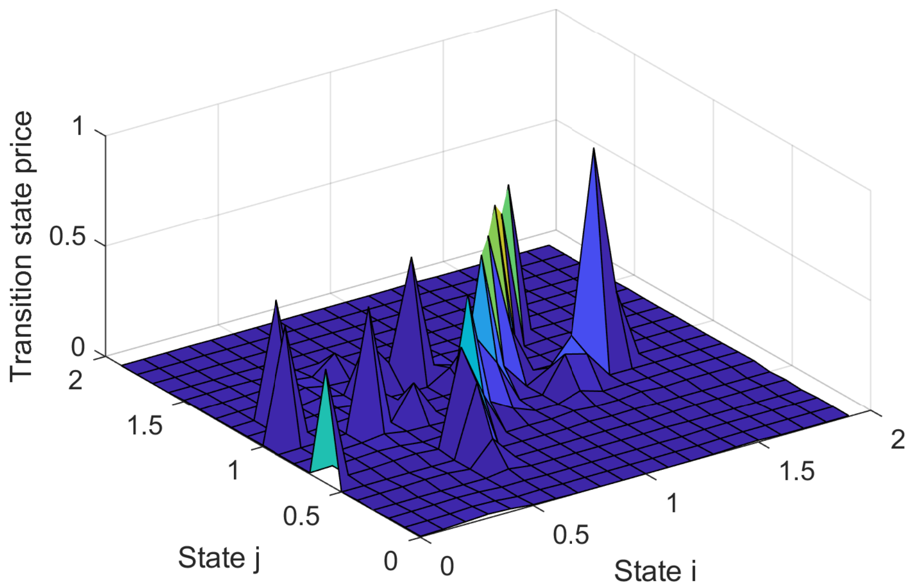
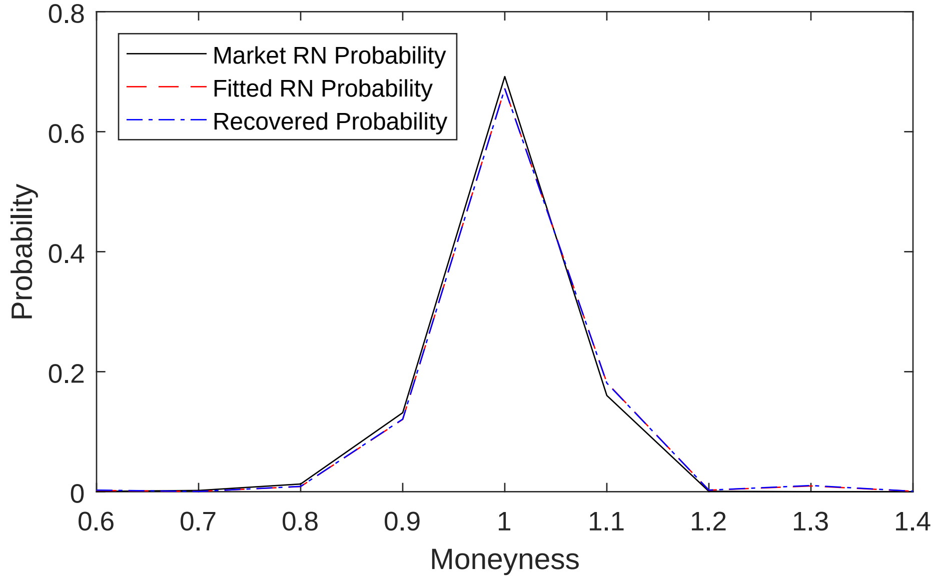
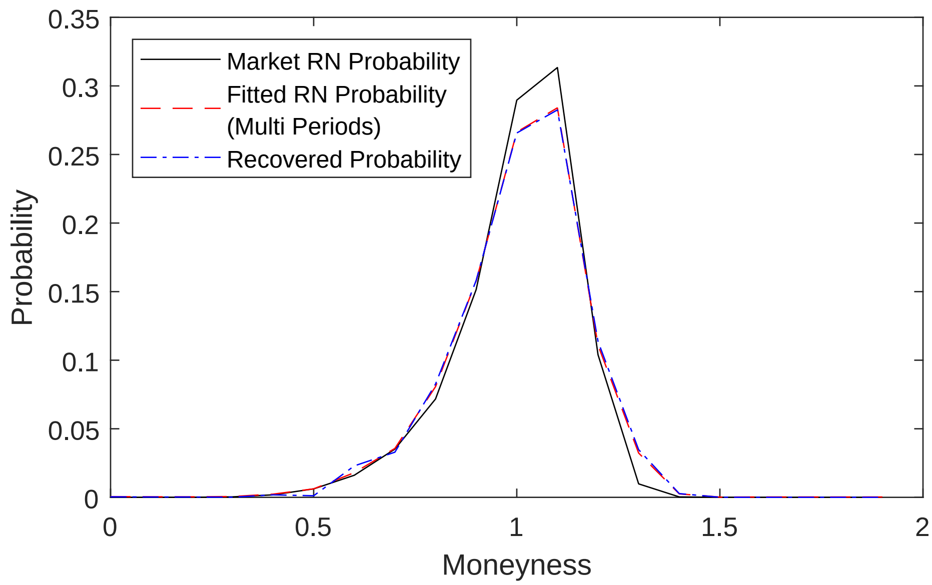


Figure 15: Ross Power Transition Matrix. This figure shows the one-period state price transition matrix based on Ross Power estimation of SPXW options on January 4, 2019. Panel B focused on the states with moneyness from 0.4 to 1.4. State i is the state at time t. State j is the state at time $t+1$.

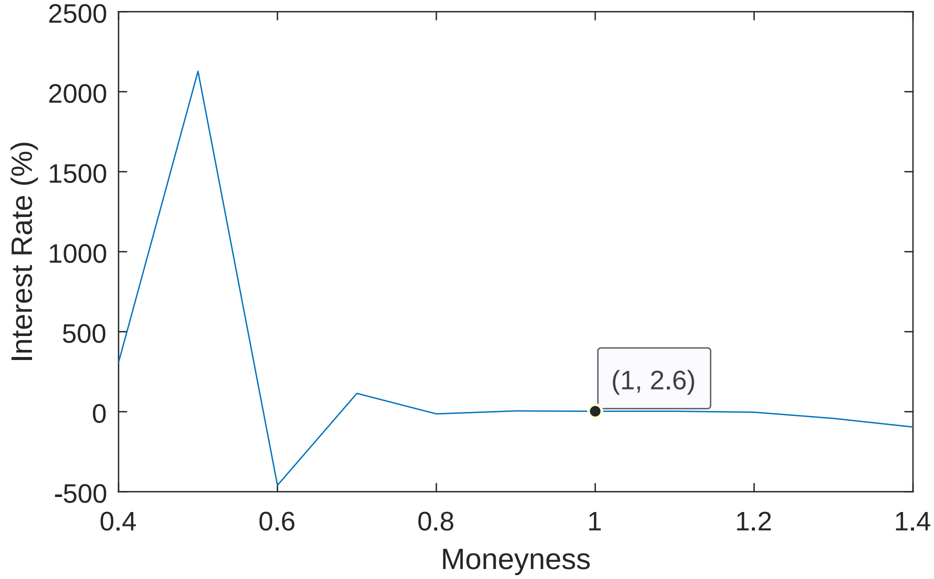


Panel A: 20-day

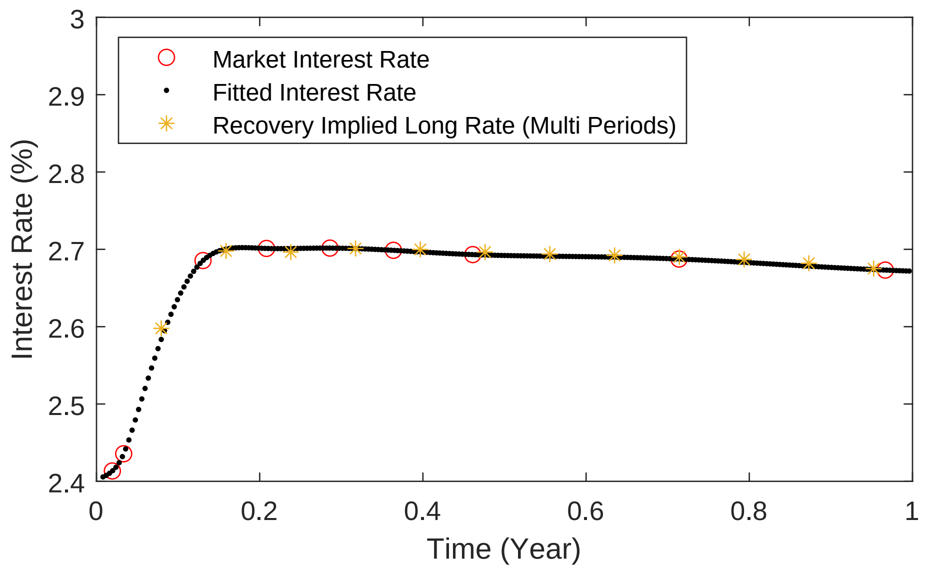


Panel B: 120-day

Figure 16: Recovered Ross Power Probability Distributions. This figure shows fitted risk-neutral (RN) probabilities and recovered probabilities based on Ross Power estimation of SPXW options on January 4, 2019. Fitted RN Probabilities (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Fitted RN Probabilities (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.



Panel A: State Interest Rate



Panel B: Long Rate Curve

Figure 17: Ross Power Implied Interest Rate. This figure shows the implied interest rate based on Ross Power estimation of SPXW options on January 4, 2019. Panel A shows the one-period state interest rates. In Panel B, Recovery implied long rates (One Period) are calculated with the market state prices at time $t-1$ and the transition matrix. Recovery implied long rates (Multi Periods) are calculated with the market state prices at t_0 and the n^{th} power of the transition matrix.

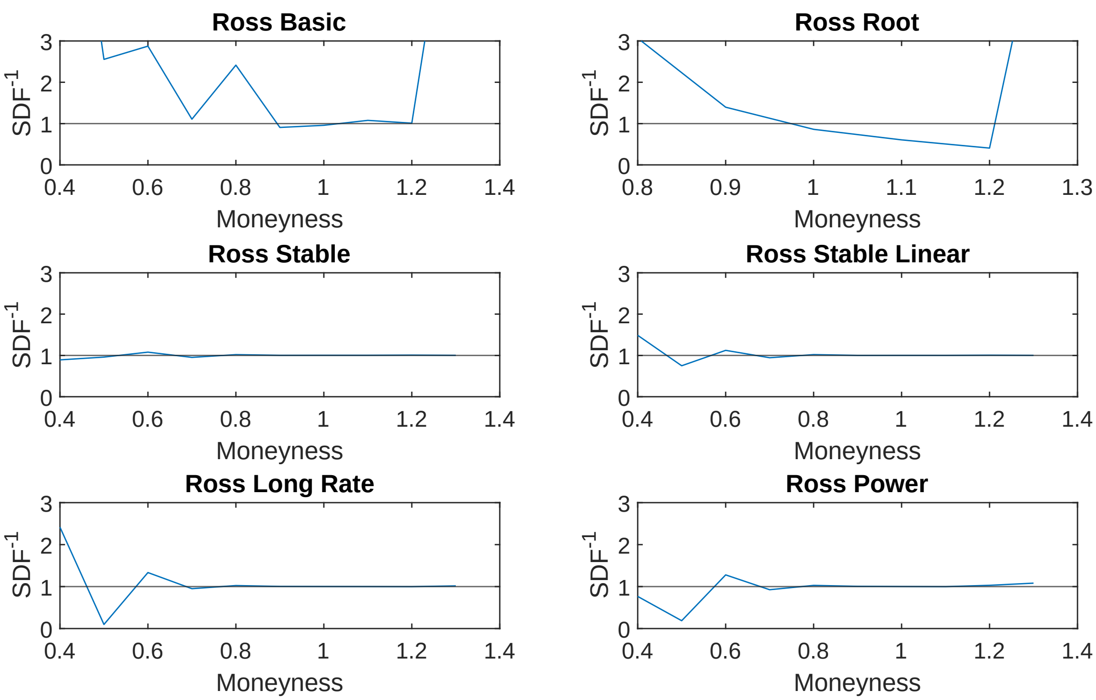


Figure 18: Stochastic Discount Factors. This figure compares the inverse of stochastic discount factor $m^{-1} = z_j/(z_0\delta)$. The horizontal line is the stochastic discount factor of a constant one.

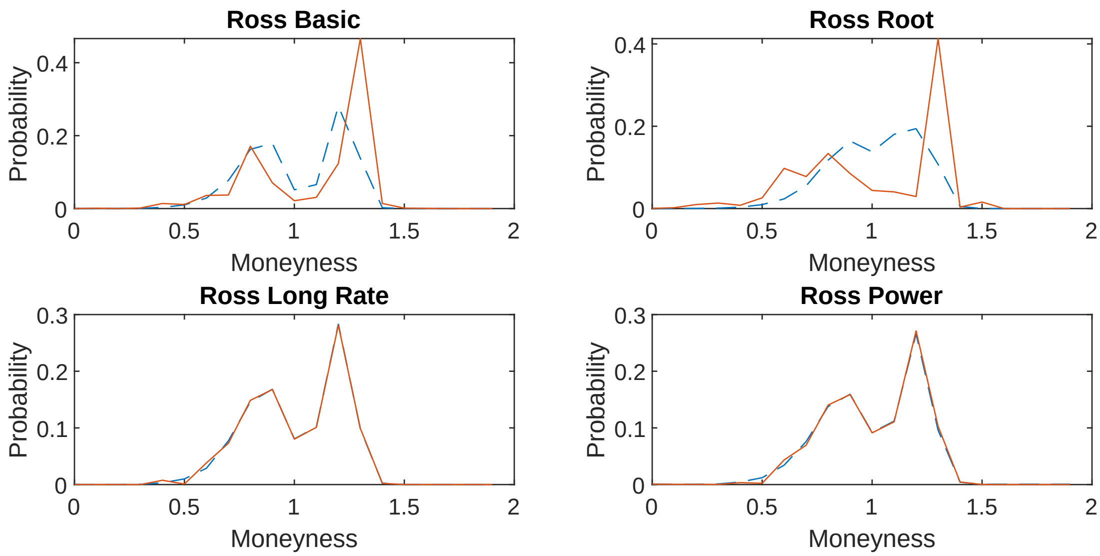


Figure 19: Long-Term Probability Distributions. This figure compares the long-term (10 years) probability distributions from different Ross recovery approaches. The dash lines are fitted risk-neutral probability distributions. The solid lines are recovered probability distributions.